Lec 22:
Graphs and Trees IV

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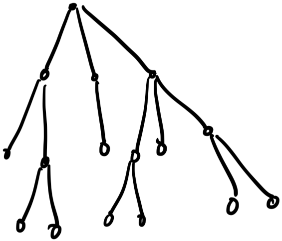
Rooted Trees

Rooted Tree

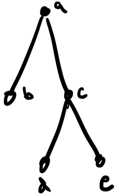
In a rooted tree there is one vertex that is distinguished from the other, the *root*. From the root vertex, all other vertices descended.



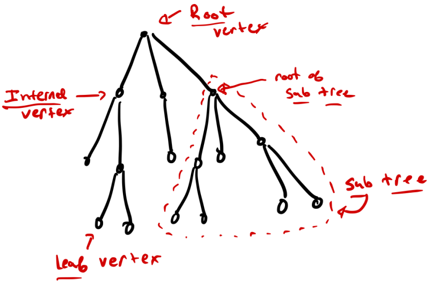
Since a Tree is acyclic, distinguishing one vertex as the root provides a way to distinguish and classify other vertices in the tree. It is also an important structure for organizing data with hierarchical relationships.

A tree is like a family

- a is the root of the tree, all vertices descend from the root
- a is the parent of b and c
- b and c are siblings
- c is the parent of d and e
- d and e are siblings
- b, c, d, e are descendants of a
- d and e are descendants of c
- b, d, and e has no descendants and thus are leafs



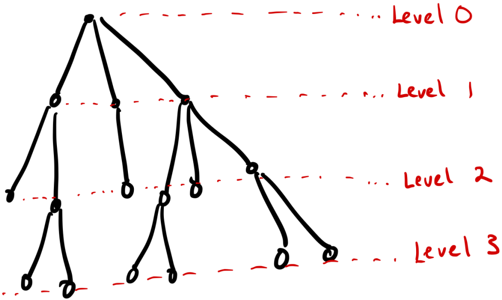
Roots, Sub-Roots, Internal and Leaf vertices



- Root of the tree has no parents
- An internal vertex is one that has a parent and a child
- A leaf vertex has a parent but no children
- An internal vertex can form the sub-root of a sub-tree

Levels and Height of a tree

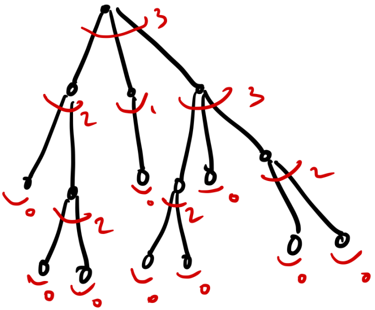
The level of the tree describes how many descendants away the given vertex is from



The root is at level 0, and each level counts down from there. The height of a tree is the maximum level of vertices in the tree. The tree above has height 3.

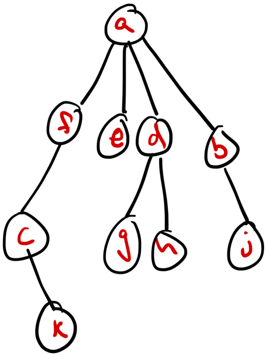
Branching Factor

The branching factor is the number of children for each parent. If a vertex has a branching factor of 0, it is a leaf node.



The branching factor of the tree is the maximum number of chidlren for each parent. The branching factor of the tree above is 3.

Exercise

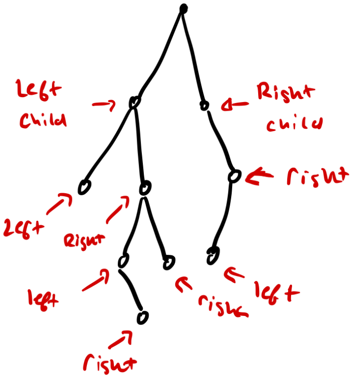


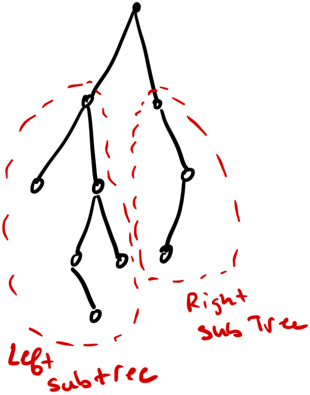
- Name all internal vertices.
- Name all leaf vertices.
- What are the siblings of f?
- What are the descendants of d?
- What is the level of j?
- What is the height of the tree?
- What is the height of the sub-tree where b is the root?
- What is the height of the sub-tree where e is the root?
- What is the branching factor of the tree?

Binary Trees

Binary Trees

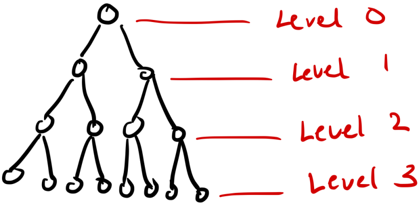
A Binary Tree is a tree where the branching factor is exactly 2. Ever internal vertex can have either a left child (or *left sub-tree*) or a right child (or *right sub-tree*)





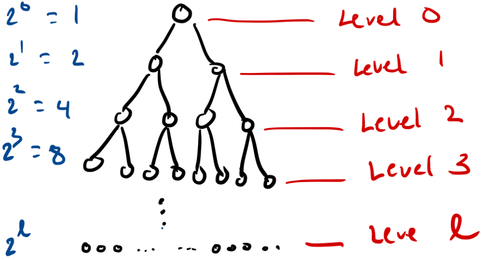
A Full Binary Tree

A full binary tree is a binary tree where every level of the tree contains the maximal number of vertices, or 0 vertices. Or, put another way, every parent (internal vertex) has exactly 2 children.



Number of Vertices in Full Binary Tree

If a tree has height h, how many vertices must be in the tree if it is a full binary tree? That is calculate $N(h)$, number of vertices for a full tree of height h, where $h \geq 0$.



$$N(h) = \sum_{\ell=0}^{h} 2^{\ell} = \frac{1 - 2^{h+1}}{1 - 2} = 2^{h+1} - 1$$

Exercise

What is the smallest number of nodes a binary tree with height h can have?

Imagine you had a full binary tree with height h, suppose at level ℓ, you select a node and remove it and all its children from the tree. How many nodes are left in the tree?

Imagine you had a full binary tree with height h. If you removed the top ℓ levels, how many full binary trees remain? And, how many nodes?

Note this exercise is new, so it is not solved in the videos.

Induction on Binary Trees

We can apply induction to prove a property of binary trees, by either inducting on the number of vertices (n) or height (h):

- Number of vertices in the tree n:
  - IH provides the property is true for all trees with n vertices, you must show it is also true with a tree with $n + 1$ vertices.
  - Note that when you remove a vertex (and edge) from a tree with $n + 1$ vertices, you have a tree with n vertices and is applicable to the IH.
  - You then need to consider the cases you can go from a tree to n vertices with $n + 1$ vertices (or vice versa) and show the property is true.
- Height of the tree h:
  - IH provides the property is true for all trees with h height, you must show it is also true with a tree with $h + 1$ vertices.
  - Note that when you remove a level from a tree with height $h + 1$, you have a tree with height h and is applicable to the IH.
  - If you remove the top level, you have a two sub-trees with height h.

Proof by induction on n (1)

Theorem

If a binary tree is full, except at the last level, and has $n \geq 1$ vertices, the height of the tree is $\lfloor \log_2(n) \rfloor$.

Base Case $P(1)$: A tree with 1 vertex has a height of 0 and $\log_2(1) = 0$

Inductive Step $P(n) \implies P(n+1)$: If a tree with n vertices where each level is full except for the last has a height of $\lfloor \log_2(n) \rfloor$, then a tree with $n+1$ vertices where each level is full except for the last has a height of $\lfloor \log_2(n+1) \rfloor$.

Consider a tree T with $n+1$ vertices. If we remove the last vertex all the way to the right on the last level, we have a tree $T^{'}$ with n vertices where everything except the last level is full. By applying the IH to $T^{'}$, we know that that $T^{'}$ has a height of $\lfloor \log_2(n) \rfloor$.

What are the ways we can go from T to $T^{'}$ by adding a vertex?

Proof by induction on n (2)

There are two cases: Either T' is a full binary tree, or T' is not a full binary tree.

- T' is a full binary tree with n vertices. It has a height $h = \lfloor \log_2(n) \rfloor$ by IH. Since T' is a full tree, adding a vertex to get T increases the height by 1, so we must show that in this case the height of T is $h + 1 = \lfloor \log_2(n+1) \rfloor$

  Consider that in T' there are $n = 2^{h+1} - 1$ vertices as it is a full tree. Adding a vertex, $n + 1 = 2^{h+1}$ thus $\log_2(n+1) = h + 1$ and $\lfloor log_2(n+1) \rfloor = h + 1$, which is what is needed to be shown.

- T' is not a full binary tree with n vertices with a $h = \lfloor \log_2(n) \rfloor$. Since it is not full tree, if we add a vertex to form T, there must be a space on the last level of T' for it and the height of T will not increase. We must show that in this case the height of T is $h = \lfloor \log_2(n+1) \rfloor$.

  Since T' is not a full binary tree $2^h - 1 < n < 2^{h+1} - 1$ because it is between a height $h - 1$ and h. When we add a vertex $2^h < n + 1 < 2^{h+1}$, or $h < \log_2(n+1) < h + 1$ when taking the log base 2. Thus $\lfloor log_2(n+1) \rfloor = h$ as the floor function rounds down to h.

QED

Proof by induction on h (1)

Theorem

If a binary tree T with height h and ℓ leaf vertices, then $\ell \leq 2^h$ (or equivalently $\log_2 \ell \leq h$).

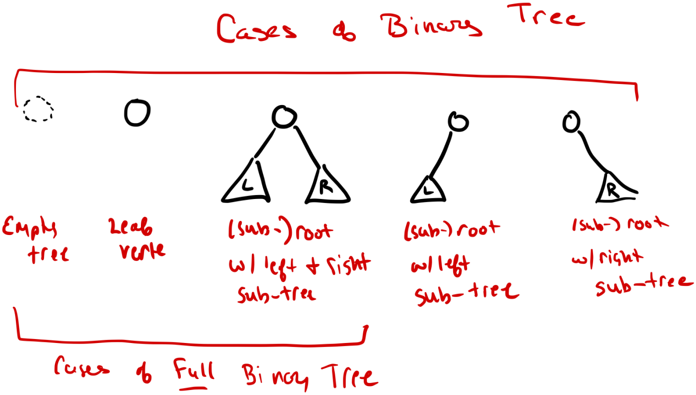
Base Case $P(0)$: A tree with height 1 has a single vertex that is a leaf vertex (and the root). So $\ell = 1$ and $\log_2(1) = 0 \leq 0$

(strong) Inductive Step $(\forall k \leq h, P(k)) \implies P(h+1)$: If a binary tree T with height $k \leq h$ has $\ell \leq 2^k$ leaf vertices, then a binary tree T' with height $h+1$ has $\ell' \leq 2^{h+1}$ leaf vertices.

If we remove the root vertex from T with height $h+1$, we are left with (potentially) two sub trees T_r and T_l each with a height $h_l \leq h$ and $h_r \leq h$. We need to consider the cases of sub trees T_r and T_l and how they would be combined to prove something about T.

Cases of Binary Trees

Every binary tree can be described in 5 cases (or 3 if it is a full tree)



Since the height is greater than 0 (proven in the base case), we have three cases. Removing the root vertex gives us a left sub tree, a right sub tree, or both.

Proof by induction on h (2)

We consider three cases when remove the root vertex.

- Case left sub-tree: We have T_l (and an empty T_r). The height of T_l is h, and by the IH, the number of leaf nodes $\ell_l \leq 2^h$. If we add back in the root to get T', we have not added any more leaf nodes so it is the case that ℓ is unchanged. Then by transitive relationship $\ell' \leq 2^h \leq 2^{h+1}$, and $\ell' \leq 2^{h+1}$ which is what was to be shown.
- Case right sub-tree: This is the same as the case above where T_l and T_r are swapped. This case is covered by the one above.

Proof by induction on h (3)

- Case right and left sub-tree: We have a T_l and T_r with heights of h_l and h_r. Since the height of T' is $h+1$, then $h+1 = \max(h_l, h_r) + 1$ because if the two sub-trees were merged at the root, then the height would increase by one more than the max height of the sub-trees. That means $h_l \leq h$ and $h_r \leq h$ and thus T_l and T_r are subject to the IH.

  The number of leaf vertices in the left sub-tree $\ell_l \leq 2^{h_l} \leq 2^h$ and right sub-tree $\ell_r \leq 2^{h_r} \leq 2^h$. If we merged the two trees into T', then the number of leafs does not change, so $\ell' = \ell_l + \ell_r$.

  As we are trying to show that $\ell' \leq 2^{h+1}$ we can consider the maximum value of $\ell_l \leq 2^h$ and $\ell_r = 2^h$. So $\ell' = \ell_l + \ell_r = 2^{h+1}$ when ℓ_l and ℓ_r are maximal, and so all other values off ℓ_l and ℓ_r would result in lesser values in their sums. Thus $\ell' \leq 2^{h+1}$. Which is what we need to show.

QED.

Exercise

Prove by induction on the height of the tree, if you had a tree T of height $h \geq 1$ and removed 1 leaf node, the height of the resulting tree is at least $h - 1$.

Note this exercise is new, so it is not solved in the videos.