Lec 21:
Graphs and Trees III

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GW

CSCI 1311 Discrete Structures I
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Tree Introduction

Acyclic Graphs

Definition

A graph is said to be circuit-free or acyclic if, and only if, it has no circuits.

A Tree is an acyclic and connected graph. A disconnected graph that is acyclic is called a forest.

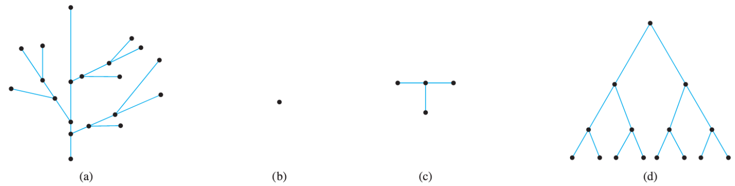
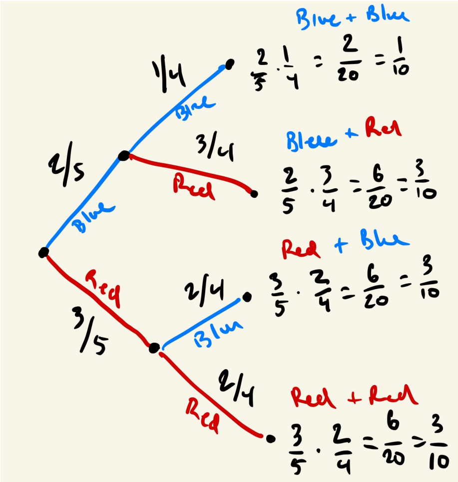


Figure 10.5.1 Trees. All the graphs in (a)–(d) are connected and circuit-free.

A vertex, by itself, is a tree, so called the trivial tree

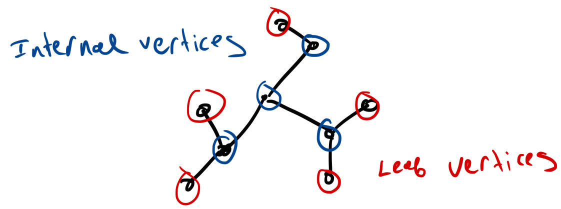
Example of Trees

The possibility/probability tree is an example tree which we've already discussed



Leafs of the Tree

A terminal vertex, or a leaf, of a tree is a vertex that has degree one.



A vertex with degree more than one, is an internal vertex

There must be leaves

Lemma

Any tree that has more than one vertex has at least one vertex of degree 1, or has at least one leaf node.

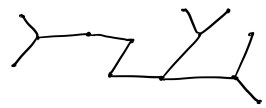
Proof by algorithm for finding a leaf

1. Choose a vertex v and an edger e incident on v.
2. Continue while $deg(v) > 1$
   - Choose e', also incident on v, and consider v' the endpoint of e'
   - Let $e = e'$ and $v = v'$.

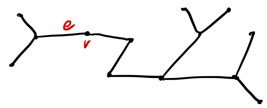
     We can never double back because a tree is acyclic
     It must terminate!
3. v must be a leaf node.

Visualizing the algorithm

Consider the Tree



Choose vertex v and e

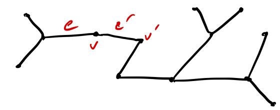


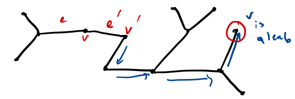
Continue while $deg(v) > 1$ (right)

...

v must be a leaf node

Consider e' and v'





... repeat

This would work, no matter the starting v and e

Number of edges in a tree

Theorem

For any positive integer n, any tree with n vertices has $n - 1$ edges.

How can we prove such a result, for all trees with n vertices? We can apply induction on trees.

What does it mean to apply induction on trees? We consider trees with vertices of increasing size. Our induction hypothesis allows us to assume the property for smaller trees.

Induction on number of vertices of a tree

The property we are trying to prove is $P(n)$

Any tree with $n \geq 1$ vertices has $n-1$ edges.

Proceed by *induction* on n:

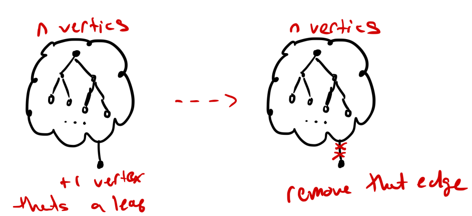
Base Case: $P(1)$ This is the trivial tree. A vertex by itself, and since there are no other vertex, it cannot have any edges because trees are acyclic.

The number of edges is $0 = 1 - 1$.

Inductive Step (1)

$P(n) \implies P(n+1)$. If we assume that a tree n vertices have $n-1$ edges (the IH), is it true that trees with $n+1$ vertices have n edges (the "to show")?

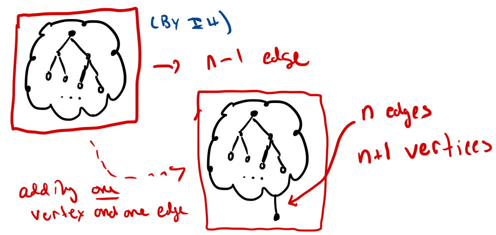
Consider a tree with $n+1$ vertices. There must be a leaf since $n \geq 1$ and thus $n+1 \geq 2$.



Find a leaf, and remove the edge and leaf vertex, giving us a tree with n edges.

Inductive Step (2)

Because the subtree, with one leaf vertex removed and the edge that connects it, has n edges, we can apply the inductive hypothesis that it must have $n-1$ edges.



Adding that vertex and edge back to any leaf will provide a tree that is acyclic. The resulting tree will have n edges (one more edge) and $n+1$ vertices. Proving our result.

Graphs and Trees

Theorem

For any positive integer n, if G is a connected graph with n vertices and $n-1$ edges, then G is a tree.

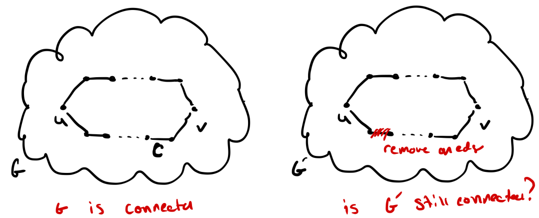
This is a much stronger theorem about the relationship between graphs and trees. First, though, we need to prove a lemma

Lemma

If G is any connected graph, C is any circuit in G, and any one of the edges of C is removed from G, then the graph remains connected.

Proof of Lemma (1)

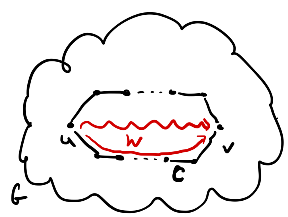
In a connected graph G with a circuit C, there would be two vertices u and v on that circuit.



If we removed an edge on that circuit, producing the subgraph G', is the graph still connected?

Proof of Lemma (2)

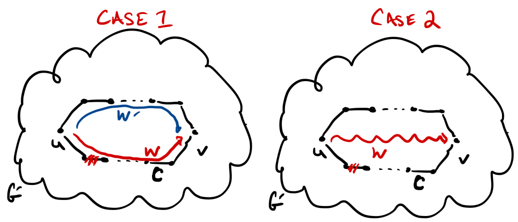
For the graph G to have been connected, there must exist a walk W between u and v (and every node).



There are two cases, was the removed edge on the walk that connected u and v or not?

Proof of Lemma (3)

Case 1: If the removed edge is on the walk, then it is also on the circuit. So we can go the "other way" around the circuit to connect u and v



Case 2: The removed edge is not on the walk. The graph is still connected.

This proves our result.

Proof of Theorem (1)

We must show that for any connected graph G with n vertices and $n-1$ edges, G is a tree.

Proof by Contradiction: Assume that G is not acyclic (thus has circuits and is not a tree).

Our goal is to show that this *cannot be the case* by deriving a contradiction, and thus the graph G is a tree since its acyclic and connected.

Proof of Theorem (2)

Assuming G has circuits. We can apply the lemma, to remove an edge from the circuit producing the connected sub-graph G'.

If G' has a circuit, we continue removing an edge from the sub-graph until we eventually reach a connected, acyclic graph G'' — that's a tree!

Since G'' has n vertices (we only removed edges), then G'' has $n-1$ edges. Then G and G'' have the same number of edges (that was part of the premise of the theorem)

BUT! To have reached G'' we had to remove edges from circuits, but G'' and G have the same number of edges — we didn't remove any edges to reach G''.

It must be the case that G didn't have cycles, thus it is acyclic and connected. It's a tree.

Exercise

Is every graph with n vertices and $n-1$ edges a tree? Provide a counter example.

Prove that if you remove an interior vertex from a tree (there are two or more edges incident on the vertex), you get a forest (a graph containing two or more trees).