Lec 20:
Graphs and Trees II

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GW

CSCI 1311 Discrete Structures I
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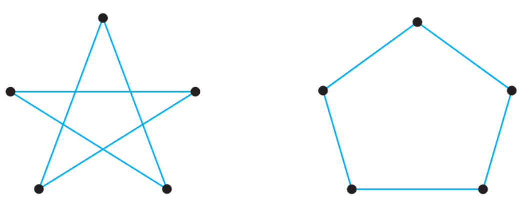
Graph Isomorphism

Recall that pictures are malleable

The following are the *same* graph



(a) (b)

Figure 10.1.1

We say that the two graphs are isomorphic.

Isomorphic Graphs

Definition

Let $G = \{E, V\}$ and $G' = \{E', V'\}$ be two graphs with edges and vertices. We say that G is isomorphic with G' if, and only if, there exists one-to-one correspondences $g : V \to V'$ and $h : E \to E'$, where h preserves the edge endpoints of E in E' based on the mapping of g.

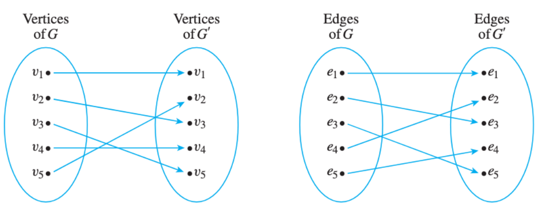
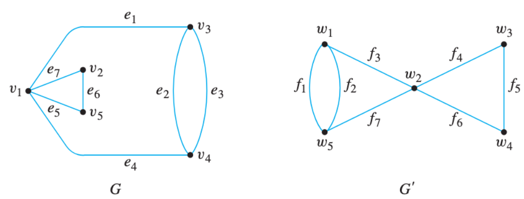


Figure 10.4.3

Exercise

Show that two graphs are isomorphic using an arrow diagram



Isomorphism is an equivalence relation

Prove it!

- Reflexive: A graph G is isomorphic to itself by using the identity function for $g : V \to V$ and $h : E \to E$.
- Symmetric: If a graph G is isomorphic to graph G', then G' is isomorphic to G. The premise provides that there must exists one-to-one correspondence g and h between G and G'. As one-to-one correspondence functions, they must have an inverse g^{-1} and h^{-1} between G' and G which are also one-to-one correspondence functions.
- Transitive: If a graph G is isomorphic to graph G', and G' is isomorphic to G'', then G is isomorphic to G''. From the premise there are one-to-one correspondences g and h from G to G', and g' and h' from G' to G''. Then the composition functions $g \circ g'$ and $h \circ h'$ are also one-to-one correspondence functions from G to G''.

Invariant of Graph Isomorphism

Definition

A property P is call an invariant for graph isomorphism if, and only if, given any graphs G and G', if G has property P and G' is isomorphic to G, then G' has property P.

How many invariant properties can you name?

Invariants

- has n vertices
- has m edges
- has a vertex of degree k
- has m vertices of degree k
- has a circuit of length k
- has a simple circuit of length k
- has m simple circuits of length k
- is connected
- has an Euler circuit
- has a Hamiltonian circuit

Matrix Representation of Graphs

Matrix (review)

Recall that a matrix is a 2-dimensional representation of a sequence. For example, a $n \times m$ matrix, A can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

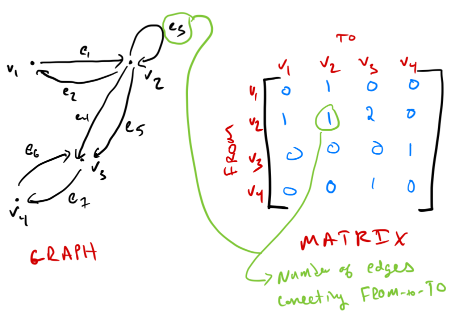
The notation a_{ij}, refers to the element at the ith row and jth column.

The ith row of the matrix is $\begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix}$

The jth column of the matrix is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$

Directed Graphs as a Matrix

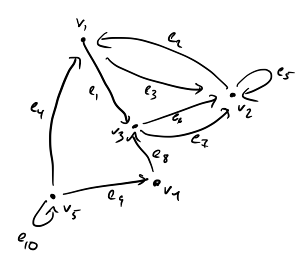
Consider the following graph, on the left.



We can write that as matrix (right) of $|V| \times |V|$, were each a_{ij} indicates the number of edges from v_i to v_j.

Exercise

Convert the following graph to a matrix.

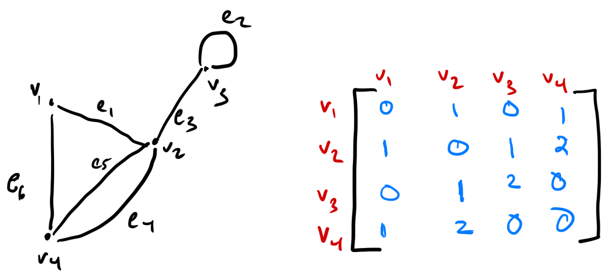


Convert the following matrix, to a graph.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 \end{bmatrix}$$

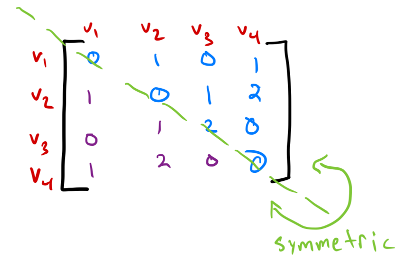
Un-directed graphs as matrix

We can use the same rules to represent an un-directed graph as a matrix



Matrix Symmetry

For a directed graph, the matrix representation is symmetric, $a_{ij} = a_{ji}$,



In an un-directed graph, an edge from v_i to v_j is also an edge from v_j to v_i

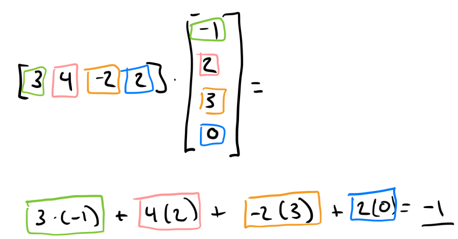
Dot Product

The scaler product or dot product of a row of matrix A with a column of matrix B, is the sum of the pairwise multiplication of each element in a row to the column.

$$\begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots a_{in}b_{n}j$$

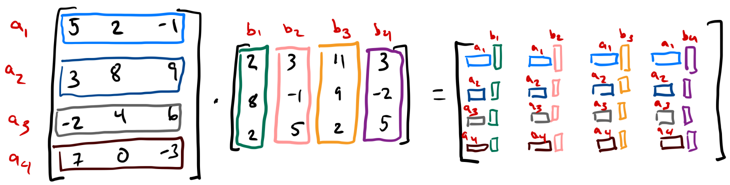
Note the number of elements in the row of A must equal the number of elements in the column of B.

Example Dot Product



Matrix Multiplication

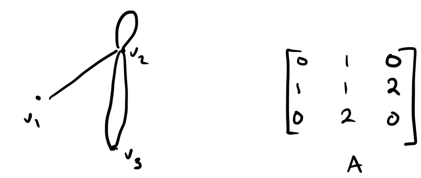
The multiplication of two matrices A and B is the row-by-column dot product.



Exercise: complete the matrix multiply above.

Graph Multiplication as way to compute walks

Consider the following graph and its matrix representation

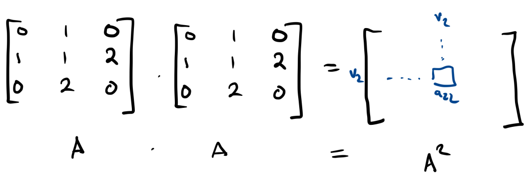


How many walks of length 1 between each node? It's encoded in the matrix!

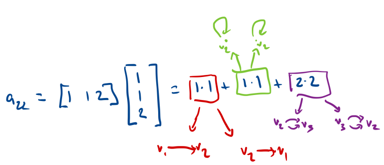
How many walks of length 2 between each node? Or circuits from v_2?

Squaring the Matrix

If we take the adjacency matric, squared. What does a value in it compute?

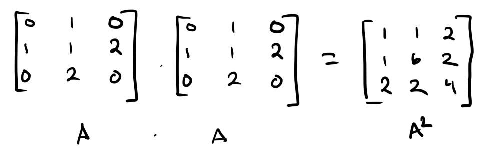


Look at a_{22}. The dot product represents the number of ways to get v_2 to another vertices multiplied by the way to get back to v_2

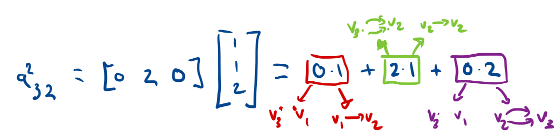


Walks of length 2

The number of walks of length 2, from v_2 and back to v_2, is $6 = a^2_{22}$.



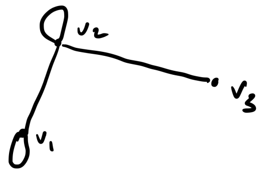
The number of walks from v_3 to v_2 of length 2, is $2 = a^2_{32}$



Go from v_3 to v_2 by either edge by one loop on v_2. There is no way to get from v_3 to either v_1 (or in reverse) in one step. So they don't count.

Exercise

How many circuits of length 3 exist in the following graph?



Recall that a circuit is a walk that begins and ends on the same vertex

Tree Introduction

Acyclic Graphs

Definition

A graph is said to be circuit-free or acyclic if, and only if, it has no circuits.

A Tree is an acyclic and connected graph. A disconnected graph that is acyclic is called a forest.

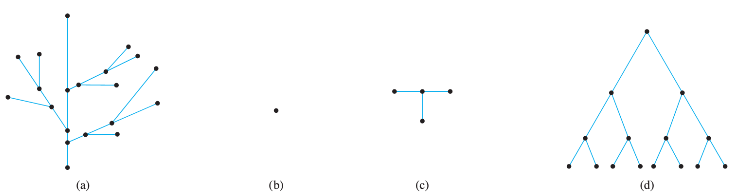
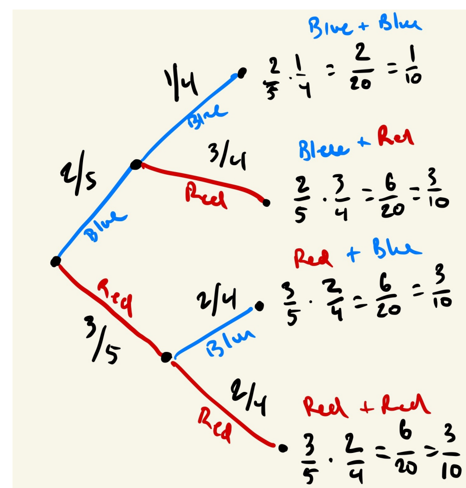


Figure 10.5.1 Trees. All the graphs in (a)–(d) are connected and circuit-free.

A vertex, by itself, is a tree, so called the trivial tree

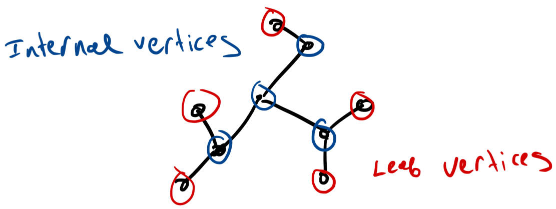
Example of Trees

The possibility/probability tree is an example tree which we've already discussed



Leafs of the Tree

A terminal vertex, or a leaf, of a tree is a vertex that has degree one.



A vertex with degree more than one, is an internal vertex

There must be leaves

Lemma

Any tree that has more than one vertex has at least one vertex of degree 1, or has at least one leaf node.

Proof by algorithm for finding a leaf

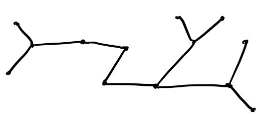
1. Choose a vertex v and an edger e incident on v.
2. Continue while $deg(v) > 1$
   - Choose e', also incident on v, and consider v' the endpoint of e'
   - Let $e = e'$ and $v = v'$.

     We can never double back because a tree is acyclic

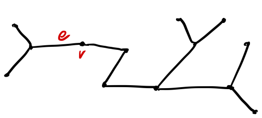
     It must terminate!
3. v must be a leaf node.

Visualizing the algorithm

Consider the Tree



Choose vertex v and e

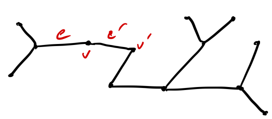


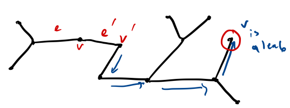
Continue while $deg(v) > 1$ (right)

...

v must be a leaf node

Consider e' and v'





... repeat

This would work, no matter the starting v and e

Number of edges in a tree

Theorem

For any positive integer n, any tree with n vertices has $n - 1$ edges.

How can we prove such a result, for all trees with n vertices? We can apply induction on trees.

What does it mean to apply induction on trees? We consider trees with vertices of increasing size. Our induction hypothesis allows us to assume the property for smaller trees.

Induction on number of vertices of a tree

The property we are trying to prove is $P(n)$

Any tree with $n \geq 1$ *vertices has* $n-1$ *edges.*

Proceed by *induction* on n:

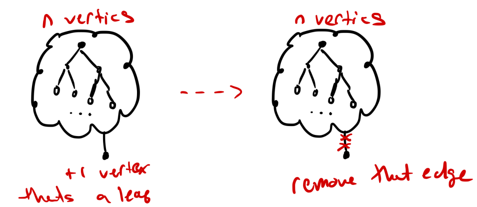
Base Case: $P(1)$ This is the trivial tree. A vertex by itself, and since there are no other vertex, it cannot have any edges because trees are acyclic.

The number of edges is $0 = 1 - 1$.

Inductive Step (1)

$P(n) \implies P(n+1)$. If we assume that a tree n vertices have $n-1$ edges (the IH), is it true that trees with $n+1$ vertices have n edges (the "to show")?

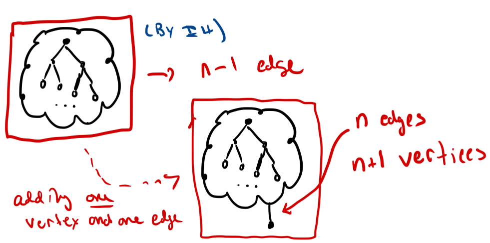
Consider a tree with $n+1$ vertices. There must be a leaf since $n \geq 1$ and thus $n+1 \geq 2$.



Find a leaf, and remove the edge and leaf vertex, giving us a tree with n edges.

Inductive Step (2)

Because the subtree, with one leaf vertex removed and the edge that connects it, has n edges, we can apply the inductive hypothesis that it must have $n-1$ edges.



Adding that vertex and edge back to any leaf will provide a tree that is acyclic. The resulting tree will have n edges (one more edge) and $n+1$ vertices. Proving our result.

Graphs and Trees

Theorem

For any positive integer n, if G is a connected graph with n vertices and $n-1$ edges, then G is a tree.

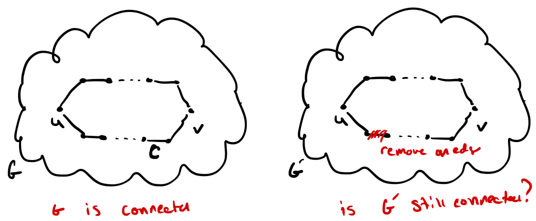
This is a much stronger theorem about the relationship between graphs and trees. First, though, we need to prove a lemma

Lemma

If G is any connected graph, C is any circuit in G, and any one of the edges of C is removed from G, then the graph remains connected.

Proof of Lemma (1)

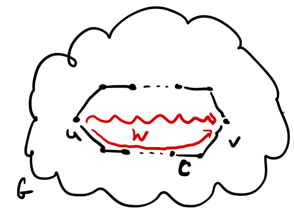
In a connected graph G with a circuit C, there would be two vertices u and v on that circuit.



If we removed an edge on that circuit, producing the subgraph G', is the graph still connected?

Proof of Lemma (2)

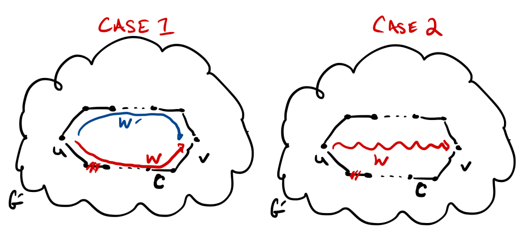
For the graph G to have been connected, there must exist a walk W between u and v (and every node).



There are two cases, was the removed edge on the walk that connected u and v or not?

Proof of Lemma (3)

Case 1: If the removed edge is on the walk, then it is also on the circuit. So we can go the "other way" around the circuit to connect u and v



Case 2: The removed edge is not on the walk. The graph is still connected.

This proves our result.

Proof of Theorem (1)

We must show that for any connected graph G with n vertices and $n - 1$ edges, G is a tree.

Proof by Contradiction: Assume that G is not acyclic (thus has circuits and is not a tree).

Our goal is to show that this *cannot be the case* by deriving a contradiction, and thus the graph G is a tree since its acyclic and connected.

Proof of Theorem (2)

Assuming G has circuits. We can apply the lemma, to remove an edge from the circuit producing the connected sub-graph G'.

If G' has a circuit, we continue removing an edge from the sub-graph until we eventually reach a connected, acyclic graph G'' — that's a tree!

Since G'' has n vertices (we only removed edges), then G'' has $n - 1$ edges. Then G and G'' have the same number of edges (that was part of the premise of the theorem)

BUT! To have reached G'' we had to remove edges from circuits, but G'' and G have the same number of edges — we didn't remove any edges to reach G''.

It must be the case that G didn't have cycles, thus it is acyclic and connected. It's a tree.

Exercise

Is every graph with n vertices and $n - 1$ edges a tree? Provide a counter example.

Prove that if you remove an interior vertex from a tree (there are two or more edges incident on the vertex), you get a forest (a graph containing two or more trees).