Lec 20:
Graphs and Trees II

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GW

CSCI 1311 Discrete Structures I
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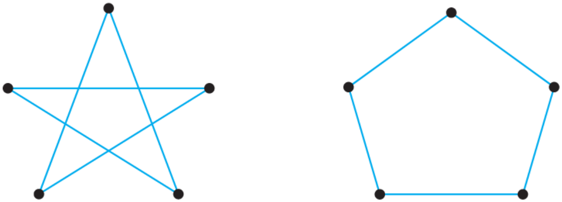
Graph Isomorphism

Recall that pictures are malleable

The following are the *same* graph



(a) (b)

Figure 10.1.1

We say that the two graphs are isomorphic.

Isomorphic Graphs

Definition

Let $G = \{E, V\}$ and $G' = \{E', V'\}$ be two graphs with edges and vertices. We say that G is isomorphic with G' if, and only if, there exists one-to-one correspondences $g : V \to V'$ and $h : E \to E'$, where h preserves the edge endpoints of E in E' based on the mapping of g.

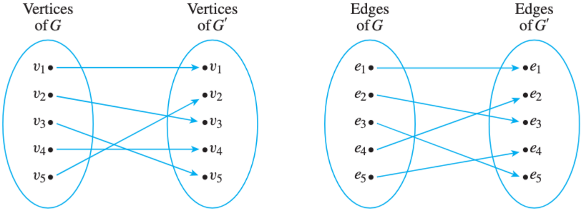
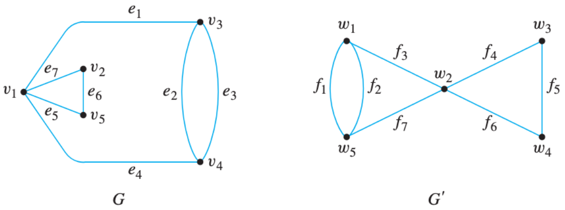


Figure 10.4.3

Exercise

Show that two graphs are isomorphic using an arrow diagram



Isomorphism is an equivalence relation

Prove it!

- Reflexive: A graph G is isomorphic to itself by using the identity function for $g : V \to V$ and $h : E \to E$.
- Symmetric: If a graph G is isomorphic to graph G', then G' is isomorphic to G. The premise provides that there must exists one-to-one correspondence g and h between G and G'. As one-to-one correspondence functions, they must have an inverse g^{-1} and h^{-1} between G' and G which are also one-to-one correspondence functions.
- Transitive: If a graph G is isomorphic to graph G', and G' is isomorphic to G'', then G is isomorphic to G''. From the premise there are one-to-one correspondences g and h from G to G', and g' and h' from G' to G''. Then the composition functions $g \circ g'$ and $h \circ h'$ are also one-to-one correspondence functions from G to G''.

Invariant of Graph Isomorphism

Definition

A property P is call an invariant for graph isomorphism if, and only if, given any graphs G and G', if G has property P and G' is isomorphic to G, then G' has property P.

How many invariant properties can you name?

Invariants

- has n vertices
- has m edges
- has a vertex of degree k
- has m vertices of degree k
- has a circuit of length k
- has a simple circuit of length k
- has m simple circuits of length k
- is connected
- has an Euler circuit
- has a Hamiltonian circuit

Matrix Representation of Graphs

Matrix (review)

Recall that a matrix is a 2-dimensional representation of a sequence. For example, a $n \times m$ matrix, A can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

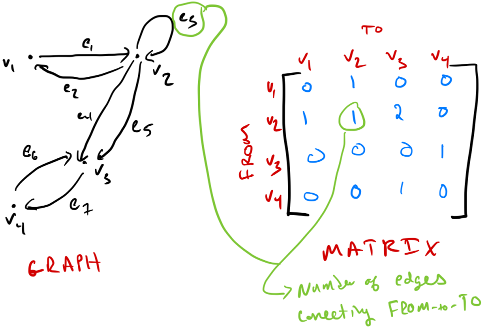
The notation a_{ij}, refers to the element at the ith row and jth column.

The ith row of the matrix is $\begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix}$

The jth column of the matrix is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$

Directed Graphs as a Matrix

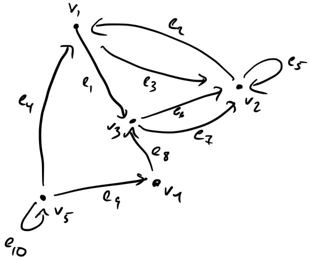
Consider the following graph, on the left.



We can write that as matrix (right) of $|V| \times |V|$, were each a_{ij} indicates the number of edges from v_i to v_j.

Exercise

Convert the following graph to a matrix.

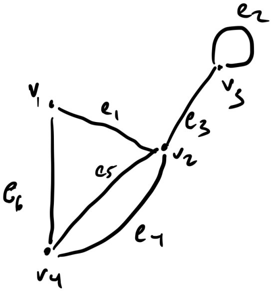


Convert the following matrix, to a graph.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 \end{bmatrix}$$

Un-directed graphs as matrix

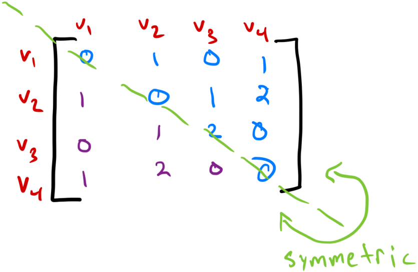
We can use the same rules to represent an un-directed graph as a matrix



	v1	v2	v3	v4
v1	0	1	0	1
v2	1	0	1	2
v3	0	1	2	0
v4	1	2	0	0

Matrix Symmetry

For a directed graph, the matrix representation is symmetric, $a_{ij} = a_{ji}$,



In an un-directed graph, an edge from v_i to v_j is also an edge from v_j to v_i

Dot Product

The scaler product or dot product of a row of matrix A with a column of matrix B, is the sum of the pairwise multiplication of each element in a row to the column.

$$\begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots a_{in}b_nj$$

Note the number of elements in the row of A must equal the number of elements in the column of B.

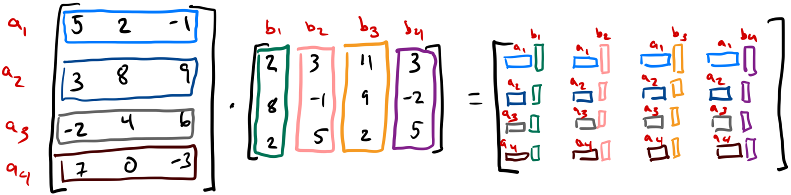
Example Dot Product

$$\begin{bmatrix} 3 & 4 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 3 \\ 0 \end{bmatrix} =$$

$$3 \cdot (-1) + 4(2) + -2(3) + 2(0) = \underline{-1}$$

Matrix Multiplication

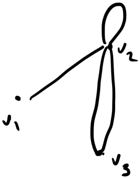
The multiplication of two matrices A and B is the row-by-column dot product.



Exercise: complete the matrix multiply above.

Graph Multiplication as way to compute walks

Consider the following graph and its matrix representation



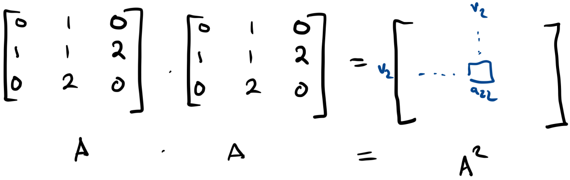
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

How many walks of length 1 between each node? It's encoded in the matrix!

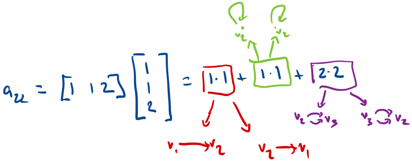
How many walks of length 2 between each node? Or circuits from v_2?

Squaring the Matrix

If we take the adjacency matric, squared. What does a value in it compute?



Look at a_{22}. The dot product represents the number of ways to get v_2 to another vertices multiplied by the way to get back to v_2



Walks of length 2

The number of walks of length 2, from v_2 and back to v_2, is $6 = a^2_{22}$.

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 6 & 2 \\ 2 & 2 & 4 \end{bmatrix}}_{A^2}$$

The number of walks from v_3 to v_2 of length 2, is $2 = a^2_{32}$

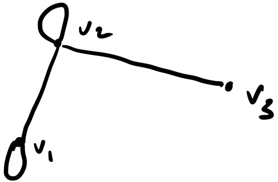
$$a^2_{32} = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 0 \cdot 1 + 2 \cdot 1 + 0 \cdot 2$$

$0 \cdot 1$: v_3: v_1, $v_1 \to v_2$; $2 \cdot 1$: $v_3 \rightrightarrows v_2$, $v_2 \to v_2$; $0 \cdot 2$: v_3: v_1, $v_2 \rightrightarrows v_3$

Go from v_3 to v_2 by either edge by one loop on v_2. There is no way to get from v_3 to either v_1 (or in reverse) in one step. So they don't count.

Exercise

How many circuits of length 3 exist in the following graph?



Recall that a circuit is a walk that begins and ends on the same vertex