Lec 19:
Graphs and Trees I

Prof. Adam J. Aviv

GW

CSCI 1311 Discrete Structures I
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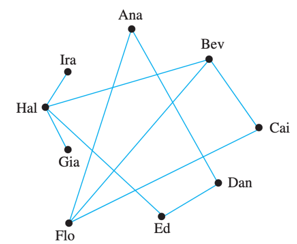
Graph Basics

Graphs

A graph is a represent information about relationships (or edges) which are represented by lines between elements (or vertices), represented by dots.

Name	Past Partners
Ana	Dan, Flo
Bev	Cai, Flo, Hal
Cai	Bev, Flo
Dan	Ana, Ed
Ed	Dan, Hal
Flo	Cai, Bev, Ana
Gia	Hal
Hal	Gia, Ed, Bev, Ira
Ira	Hal



For example, the above graph represents students in a class, and the edges represent a relationship between students who have previously worked together on projects.

Graphs Formally

Definition

A graph $G = \{V, E\}$ consists of two finite sets, the set of vertices V and the set of edges E.

For all $e \in E$, $e = (v_i, v_j)$ where $v_i \in V$ and $v_j \in V$, and v_i and v_j are the endpoints of the edge. An edge connects its endpoints.

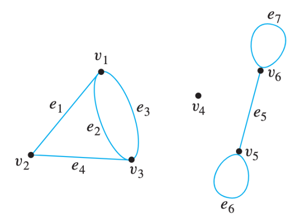
If two vertices are endpoints of an edge, the vertices are adjacent vertexes. Two edges that share an endpoint, or are incident on a vertex, are adjacent edges.

If an edge has the same endpoints, that is $v_i = v_j$, then the edge is a loop and is adjacent to itself.

To edges, say $e_1 = (v_i, v_j)$ and $e_2 = (v_k, v_l)$, are parallel if they have the same end points $v_i = v_k \wedge v_j = v_l$ (or $v_j = v_k \wedge v_i = v_l$)

Exercise

For the following graph,



Which edges are loops?

Which edges are parallel?

What is the set of edges and vertices, and what is the set of endpoints?

Pictures are malleable

The following are the *same* graph

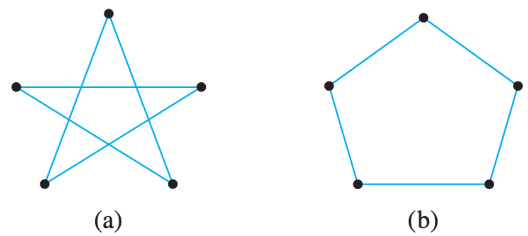


Figure 10.1.1

How could we show that?

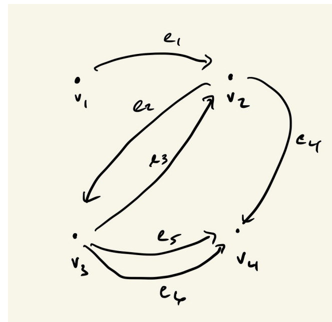
We can find a labeling of the vertices and edges that are the same between the graphs, such that all the endpoints are the same for the same set of vertices.

Directed Graphs

Definition

A directed graph is a graph where the order of the endpoints for an edge matter. An edge $e \in E$, $e = (v_i, v_j)$ describes an edge *directed* from v_i to v_j, which is different and not parallel to the edge $f \in E$, $f = (v_j, v_i)$.

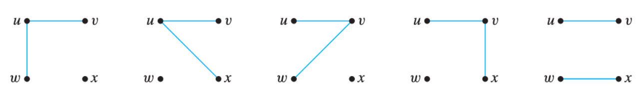
We typically use arrows for edges in directed graphs.



Simple and Complete Graphs

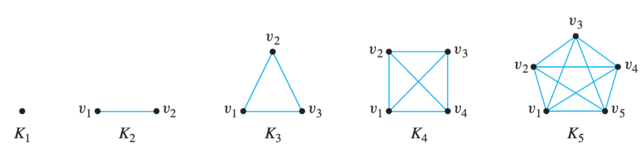
Definition

A simple graphs is a graph that does not have any loops or parallel edges.



Definition

A complete graph of n vertices denoted K_n, is a simple graph with n vertices and exactly one edge connecting each pair of distinct vertices

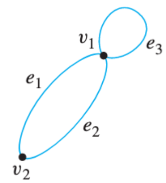


Subgraphs

Definition

A graph H is a subgraph of a graph G if, and only if, every vertex in H is also a vertex in G, every edge in H is also an edge in G, and ever edge in H has the same endpoints as it has in G.

What are all the subgraphs of the following graph?



Graph Degree

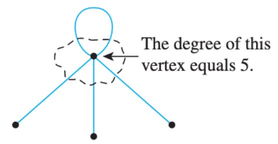
Degree of a vertex

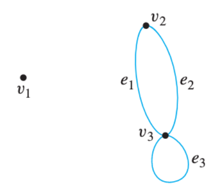
Definition

The degree of a vertex v, denoted $deg(v)$, equals the number of edges that are incident on v, with an edge that is loop counted twice.

The total degree of G is the sum of the degrees of all the vertices of G.

What is the degree of each of the vertexes in the following graph?





Handshake theorem

Handshake theorem

If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges in G

Why would this be true?

Proof.

For any edge in the graph, it is either a loop or a non-loop. If it is a loop it contributes 1 to each vertex it connects, and if it is a loop, it contributes 2. In either case, each edge contributes 2 degree. Thus the total degree of a graph is equal to twice the number of edges. □

Corollary

The total degree of a graph is even.

Exercise

Is it possible to have a graph with vertices of degree 1, 1, 2, and 3.

Draw a simple graph with vertices of degree 1, 1, 2, 2, 1, and 1.

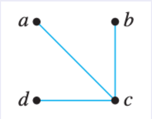
Can you draw a simple graph with vertices of degree 1, 1, 3, and 3

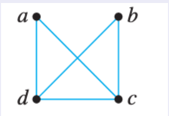
Proving a simple graph cannot exist

There does not exist a simple graph with vertices of degree 1, 1, 3, and 3

Proof.

By contradiction, assume such a graph exists. It has four vertices: a, b, c, and d. Because two of the vertices have degree 3, lets say $deg(c) = 3$, then there must exists edges from c to a, b, and d since the graph is simple and does not contain loops.





By the same logic, a second vertex, say d, would also need to connect to all three nodes. Now the $deg(a) \geq 2$ and $deg(b) \geq 2$, contradicting the premise. □

Even number of odd-degree nodes

In any graph, there is always an even number of numbers with odd degree.

Proof.

Consider a graph G with n nodes of even degree $n > 0$ and m nodes of odd degree $m \geq 0$. Let E be the sum of the degree of the even degree nodes, and O be the sum of the degree of the odd degree nodes. Then the total degree T is the sum of E and 0

$$T = E + 0$$
$$O = T - E$$

Since T ad E must be even, and O is the difference of two even numbers, then O is even. The degree of each odd node is odd, so there must be even number of them if O is even because the sum of an odd number of odd numbers is odd. □

Exercises

Does there exists a simple graph where each vertex has even degree? Where each vertex has odd degree?

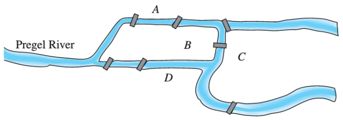
Recall that K_n refers to a complete graph with n vertices, show that the number of edges of $edges(K_n) = \frac{n(n-1)}{2}$

Consider an acquaintance graph, where edges describe that two individuals (vertices) are friends. Is it possible in a group of 9 people for each to be friends with exactly five others?

Trails, Paths, and Circuits

Euler and Königsberg

The town of Königsberg in Prussia (now Kaliningradin Russia) was built at a point where two branches of the Pregel River came together. It consisted of an island and some land along the river banks. Is it possible for a person to take a walk around town,starting and ending at the same location and crossing each of the seven bridges exactly once?



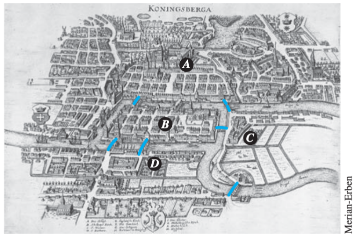


Figure 10.2.1 The Seven Bridges of Königsberg

Graph of Königsberg Map

Try it yourself, can you find a *walk* where you do not use a bridge/edge more than once?

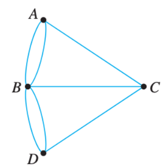
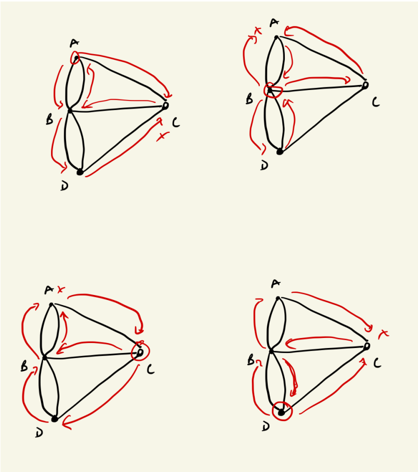
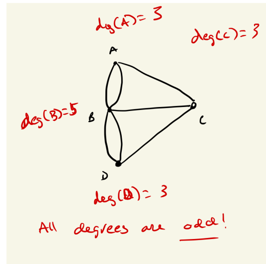


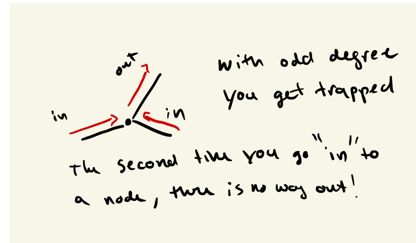
Figure 10.2.2 Graph Version of Königsberg Map

All attempts fail



An issue of degree





All degrees of Königsberg graph are odd, there must be a place you get stuck!

Walks formally

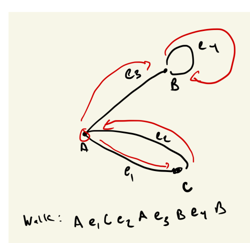
Definition

A walk from v to w is a finite alternating sequence of adjacent vertices and edges of G, of the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n$$

where v's represent vertices, the e's represent edges, where for all i, v_{i-1} and v_i are endpoints for edge e_i.

Sample Walk:



Trails, Paths, Circuits

- A trail from v to w is a walk from v to w that does not contain A repeated edge.
- A path from v to w is a trail that does not contain a repeated vertex.
- A closed walk is a walk that starts and ends at the same vertex.
- A circuit is a closed walk that contains at least one edge and does not contain a repeated edge.
- A simple circuit is a circuit that does not have any other repeated vertex except the first and last.

Connectedness

Definition

Let G be a graph.

- Two vertices v and w of G are connected if, and only if, there is a walk from v to w.
- The graph G is connected if, and only if, given *any* two vertices v and w, there is a walk from v to w.

$$G \text{ is connected} \iff (\forall v, w \in V(G))(\exists \text{ a walk from } v \text{ to } w)$$

Euler Circuits

Returning to the Königsberg bridges ...

Definition

Let G be a graph. An Euler circuit for G is a circuit that contains every vertex and every edge of G, at least once.

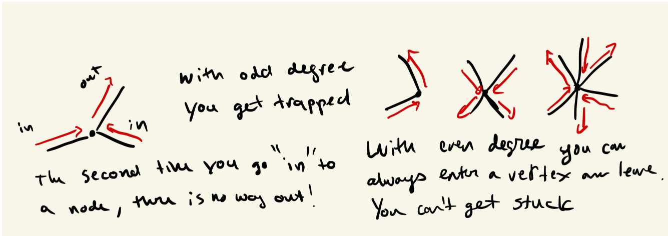
Recall that a circuit is a walk that starts and ends at the same point and does not repeat an edge. So an Euler Circuit:

- Starts and ends at the same point
- Does not repeat an edge
- Visits every vertex

Finding a Euler circuit

Theorem: Even Degrees of Eueler Circuit

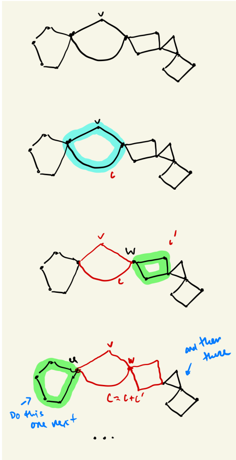
If a graph G is connected and the degree of every vertex of G is a positive even integer, than G has an Euler Circuit



We will prove this by algorithm! Or, by providing a process to always find an Euler circuit in the connected, positive-even-degree graph G.

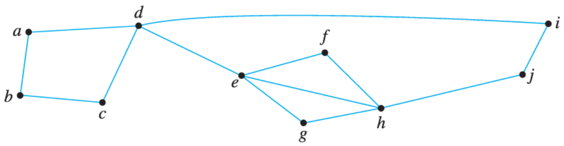
Algorithm for Finding an Euler Circuit

1. Pick any vertex v of G to start
2. Pick any sequence of adjacent vertices and edges starting and ending with v, call this circuit C.
3. Check whether C is an Euler Circuit? If so, return C
   - Generate a subgraph G' by removing all edges in C from G and any isolated vertices
   - Pick a vertex w from C that is in G'
   - Pick any sequence of adjacent vertices and edges starting and ending with w, call this circuit C'
   - Merge circuits C and C' into a new circuit C'', where you perform circuit C starting with w following by circuit C'.
   - Assign $C = C''$ and proceed to Step 3.



Exercise

Use the algorithm to find an Euler Circuit in the following graph

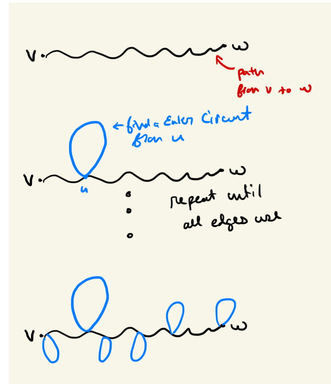


Euler Trails

Like an Euler Circuit, we can instead consider Euler Trails where we start at a vertex v and end at a vertex w, while traversing every edge exactly once.

Corollary

Let G be a graph, and let v and w be two distinct vertices of G. There is a Euler trail from v to w if, and only if, G is connected, v and w have odd degree, and all other vertices have positive degree



Hamiltonian Circuits

Definition

Given a graph G, a Hamiltonioan circuit for G is a simple circuit that includes every vertex of G. That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, except for the first and last, which are the same.

Euler Circuit

- Visit every edge exactly once.
- Visit every vertex at least once.

Hamiltonian Circuit

- Visit every vertex exactly once.
- Not all edges included.

Graph Properties with a Hamiltonian Circuit

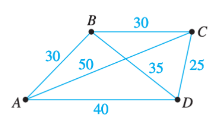
Claim: If a graph G has a Hamiltonian circuit, than G has a subgraph H with the following properties

1. H contains every vertex of G
2. H is connected
3. H has the same number of edges as vertices
4. Every vertex of H has a degree 2

Why must these things be true?

Traveling Salesman Problem

Imagine a salesman that needs to travel between n cities without visiting a city twice and minimizing the distance travel while ending the journey back where it started. Which path should the salesman take?



Route	Total Distance (In Kilometers)	
$ABCDA$	$30+30+25+40=125$	
$ABDCA$	$30+35+25+50=140$	
$ACBDA$	$50+30+35+40=155$	
$ACDBA$	140	[$ABDCA$ backwards]
$ADBCA$	155	[$ACBDA$ backwards]
$ADCBA$	125	[$ABCDA$ backwards]

Enumerate all Hamiltonian Circuits and choosing the smallest is still the best known algorithm to get a correct result[1]. An important, well known *hard* problem in computer science.

[1] There are good approximation algorithms