Lec 18:
Prob. and Counting III

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GW

CSCI 1311 Discrete Structures I
Spring 2023

Birthday Paradox

The birthday paradox

There are 22 players on a football field, 11 a side, what is the probability that at least two players share the same birthday?

Can we solve a related problem, what is the probability that no players out of the 22 share the same birthday?

Birthday Paradox (2)

Let's consider the probabilities as we add people to the "pitch". Let $\bar{p}(n)$ be the probability that the n-th player added to the pitch does not share a birthday with any previous player:

- Adding one person: the probability of not sharing a birthday is 1.
  - $\bar{p}(1) = 365/365 = 1$
- Adding second person: there is 364 days for person-2's birthday, and person-1 does not share a birthday.
  - $\bar{p}(2) = 364/365 \cdot \bar{p}(1) = 364/365 \cdot 365/365$
- Adding third person: there is 363 days where person-3 doesn't share a birthday with person-2 and person-1, and person-1 and pserson-2 do not share a birthday.
  - $\bar{p}(3) = 363/365 \cdot \bar{p}(2) = 363/365 \cdot 364/364 \cdot 365/365$
- Adding fourth person: there is 362 days wehre person-4 doesn't sahre a birthday with person-3,-2,-1, and person-3,-2,-1 do not share a birthday.
  - $\bar{p}(4) = 362/364 \cdot \bar{p}(3) = 362/365 \cdot 363/365 \cdot 364/365 \cdot 365/365$
- ...

Birthday Paradox (3)

Iterating, we are left with the following calculation

$$\bar{p}(22) = (1/365)^{22}(364 \cdot 363 \cdot \ldots \cdot 344) = 0.526$$

Then $p(n) = 1 - \bar{p}(n)$ is the probability that at least one player shares a birthday with another: $p(22) = 1 - 0.526 = 0.474 = 47.4\%$.

Generalizing Birthday Paradox

Consider that $\bar{p}(n)$ can be rewritten as:

$$\begin{aligned}\bar{p}(n) &= 365/365 \cdot 364/364 \cdot \ldots (365 - (n-1))/365 \\ &= \frac{365 \cdot 364 \cdot \ldots (365 - n + 1)}{365^n} \\ &= \frac{P(365, n)}{365^n} \qquad \textit{P(r,s) is permutation}\end{aligned}$$

Or, how many ways can we arrange n unique birthdays, over the total sample space of all possible n (non-unique birthdays).

Exercise

Suppose you are drawing numbers at random between 1 and 10.

After drawing 2 numbers, what is the probability that you've selected two numbers that are the same?

After drawing 5 numbers, what is the probability that you've selected two numbers that are the same?

After how many number draws would you be certain that at least two of the numbers are the same?

Conditional Probability

Boy/Girl Problem

Suppose you know a couple with two children. If you knew that at least one of the children was a boy, what is the probability that other child is also a boy?

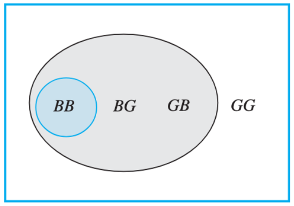


Figure 9.9.1

Boy/Girl Problem (2)

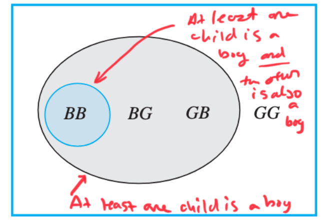


Figure 9.9.1

$$\frac{P(\text{at least one child is a boy } \textbf{and} \text{ the other child is a boy})}{P(\text{at least one child is a boy})}$$

There are 3/4 cases that at least one child is a boy, of which 1/4 both are boys.

$$\frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Conditional Probability

Another way we can frame the Boy/Girl probability is as a conditional probability statement, of events A and B

$\boxed{P(B \mid A)}$ read, probability of B given A

Where A is the probability that at least one child is a boy, and B is the probability that the other child is a boy. We solve for the conditional by considering the sample space when $P(A)$ is so, and the events when $P(A \cap B)$ is so.

$$\boxed{P(B \mid A) = \frac{P(B \cap A)}{P(A)}}$$

Of course, if $P(A) = 0$, then $P(B \mid A) = 0$ since it could never be the case that "given A."

Exercise

Suppose you have two (fair) dice. You roll them . . .

What is the probability that the sum of the dice is 10, if one of the dice is showing a 4?

What is the probability that the sum of the dice is 7, if one of the dice is showing an odd number?

What is the probability that the sum of the dice is less than or equal to 6, if the values on both of the dice is even?

More Exercises!

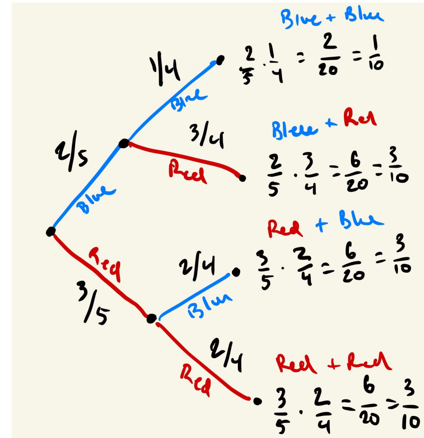
Consider an urn with 3 red balls and 2 blue balls

If you draw two balls from the urn, what is the probability that the two balls are red?

If you draw two balls from the urn, if at least one of the balls was red, what is the probability that the other ball is blue?

Draw a tree diagram of the possible outcomes for drawing two balls with the probabilities.

Tree Diagram (Answer)



Bayes' Rule

More Urns!

Consider that there are two urns, U_1 and U_2. In U_1 there are 3 red and 4 blue balls, and in U_2 there are 3 red and 5 blue balls.

To draw a ball, you first flip a coin. If it is heads, select a ball from U_1, otherwise select a ball from U_2 if it is tails.

If the ball drawn was blue, what is the probability it was drawn from U_1?

Conditionals of Two Urns

Let's formalize the problem by calling the event of drawing a blue ball as A. Our goal is to find:

$$P(U_1 \mid A) = \frac{P(U_1 \cap A)}{P(A)}$$

Note that it is relatively easy for us to find these two conditionals

$$\underbrace{P(A \mid U_1) = \frac{4}{7}}_{\text{4-of-7 balls in the Urn-1 are blue}} \qquad \underbrace{P(A \mid U_2) = \frac{5}{8}}_{\text{5-of-8 balls in Urn-2 are blue}}$$

But to solve $P(U_1 \mid A)$ we need to know $P(A)$ and $P(U_1 \cap A)$.

Conditional Probabilities Rearranged

Recall the formula for conditional probability gives us:

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

By multiplying both sides by $P(A)$ we can derive the formula for the intersection probability

$$P(A) \cdot P(B \mid A) = P(B \cap A)$$

How can that help with our two urn problem?

Finding $P(U_1 \cap A)$ and $P(U_2 \cap A)$

The question is: *If you draw a blue ball, what is the probability it came from the first urn?*

$$P(U_1 \cap A) = P(A \cap U_1) = P(U_1) \cdot P(A \mid U_1)$$

$P(U_1) = 1/2$ and $P(A \mid U_1) = 3/7$. So we can solve this directly:

$$P(U_1 \cap A) = \frac{1}{2} \cdot \frac{3}{7} = \frac{3}{14} \qquad P(U_2 \cap A) = \frac{1}{2} \cdot \frac{5}{8} = \frac{5}{16}$$

How does this help us find $P(A)$?

Solving for $P(A)$

Note that the events $(U_1 \cap A)$ and $(U_2 \cap A)$ are disjoint, in that it is impossible to draw from both urns. Or put another way

$$(U_1 \cap A) \cap (U_2 \cap A) = \emptyset$$

Also note, that the event A either occurred in the sets $(U_1 \cap A)$ or $(U_2 \cap A)$. So:

$$P(A) = P((A \cap U_1) \cup (A \cap U_2))$$

When two events are disjoint, and we do not have to subtract the intersection when considering the union, leading to the direct calculation:

$$\begin{aligned} P(A) &= P((A \cap U_1) \cup (A \cap U_2)) \\ &= P(A \cap U_1) + P(A \cap U_2) \\ &= \frac{3}{14} + \frac{5}{16} = \frac{59}{112} \end{aligned}$$

Solving $P(U_1 \mid A)$

We can now complete our calculations:

$$P(U_1 \mid A) = \frac{P(U_1 \cap A)}{P(A)}$$

$$= \frac{\frac{3}{7}}{\frac{59}{112}}$$

$$= 40.7\% \text{ or so}$$

Exercise

Draw a tree diagram from the previous two-urn problem for all possible outcomes of drawing a ball: Urn-1 has 3 blue and 4 red balls, and Urn-2 has 5 blue and 3 red balls.

What is the probability of drawing a red ball?

If you drew a red ball, what is the probability it came from Urn-2?

Bayes Theorem

Bayes' Theorem

Suppose that a sample space S is a union of mutually disjoint events $B_1, B_2, B_3, \ldots, B_n$, suppose A is an event in S, and suppose A and all the B_i have nonzero probabilities. if k is an integer with $1 \leq K \leq n$, then

$$P(B_k \mid A) = \frac{P(A \mid B_k)P(B_k)}{P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \cdots + P(A \mid B_n)P(B_n)}$$

Example: With the two-urns, we had mutually disjoint events of drawing from Urn-1 and Urn-2 (these would be U_1 and U_2), and another event A, drawing a blue ball. Then

$$P(U_1 \mid A) = \frac{P(A \cap U_1)}{P(A)} = \frac{\overbrace{P(A \mid U_1)P(U_1)}^{P(A \cap U_1)}}{\underbrace{\overbrace{P(A \mid U_1)P(U_1)}^{P(A \cap U_1)} + \overbrace{P(A \mid U_2)P(U_2)}^{P(A \cap U_2)}}_{P(A)}}$$

Exercise

You have six dice, 3 red, 2 blue, and 1 yellow, choose one at random to roll.

Given that you rolled a 6, what is the probability it was on a red dice?

Given that you rolled a 1, what is the probability it was on a yellow dice?

False Positives/Negatives and True Positives/Negatives

Consider a medical test for a medical condition. Every person either has or does not have the condition, and the test either produces a yes or no for each person.

- **True-Positive**: The person *has* the condition, and the test returns *yes*.
- **True-Negative**: The person *does not have* the condition, and test returns *no*.
- **False-Positive**: The person *does not have* the condition, and the test returns *yes*.
- **False-Negative**: The person *has* the condition, and the test returns *no*.

Positive Test Results

Suppose a medical condition is found in 5 of 1,000 people. The test for the condition has a false positive rate of 3% and a false negative rate of 1%.

What is the probability that a randomly chosen person who tests positive for the condition actually has the condition?

Formalizing Positive Test Result

Let B is the event of having the condition, and B^c is not.

$$P(B) = 5/1000 = 0.005 \qquad P(B^c) = 1 - P(A) = 0.995$$

Let A be the event of the test testing positive, and A^c is testing negative.

$$\overbrace{P(A \mid B)}^{\text{True-Positive}} = 0.99 \qquad \overbrace{P(A^c \mid B)}^{\text{False-Negative}} = 0.01$$

$$\underbrace{P(A \mid B^c)}_{\text{False-Positive}} = 0.03 \qquad \underbrace{P(A^c \mid B^c)}_{\text{True-Negative}} = 0.97$$

"The test for the condition has a false positive rate of 3% and a false negative rate of 1%."

Test positive and actually have the condition?

What is the probability that a randomly choose person who tests positives, actually has the condition?

$P(B \mid A)$ = "Probability of having the condition, given a positive test"

By Bayes' Rule

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A \mid B)P(B) + P(A \mid B^c)P(B^c)}$$

$$= \frac{(0.99)(0.005)}{(0.99)(0.005) + (0.03)(0.995)}$$

$$\approx 0.1422 \approx 14.2\%$$

Exercise

What is the probability that a randomly chosen person who tests negative for the condition does not indeed have the disease?