

Permutations

Definition

A permutation of a subset of objects is the number of ways those objects can be arranged in a row.

Example

If there are 6 diplomats, and 4 seats around a table, how many different permutations are there for 6 diplomats to sit at the table where two diplomats stand?

$$6 \cdot 5 \cdot 4 \cdot 3 = 6!/2! = 360$$

6 diplomatic in the first seat, 5 can be selected for the second seat, 4 in the third, and 3 in the fourth.

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r-Permutation

Definition

An *r*-permutation of a set of *n* elements is an ordered selection of *r* elements take from the set of *n* elements. The number of *r*-permutations of a set of *n* elements is denoted as P(n, r) or n P r, read "*n* permute *r*."

If *n* and *r* are integers and $1 \le r \le n$, the number of *r*-permutations of a set *n* elements is given by the formula:

$$P(n,r) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)$$

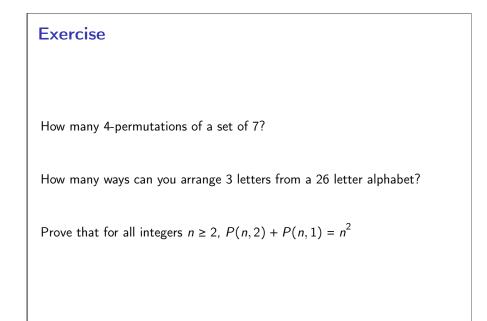
or equivalently

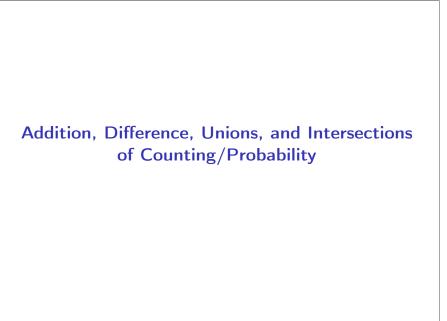
$$P(n,r) = \frac{n!}{(n-r)!}$$

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Example: PINs of multiple lengths

Suppose you want to count the number of 1-, 2-, 3-, and 4-length PINs selected from digits 0-9 without repetition?

Calculation at each length:

- 1-length: P(10, 1) = 10 = 10
- 2-length: $P(10, 2) = 10 \cdot 9 = 90$
- 3-length: $P(10,3) = 10 \cdot 9 \cdot 8 = 720$
- 4-length: $P(10, 4) = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$

The total PINs of these lengths is the sum, 10 + 90 + 720 + 5040 = 5860

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Exercise

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Addition Rule for Disjoint Sets

Addition Rule

Suppse a finite set A equals the union of k distinct mutually disjoint subsets A_1, A_2, \ldots, A_k , Then

 $N(A) = N(A_1) + N(A_2) + \ldots + N(A_k)$

Example: PINS of different lengths are disjoint sets. So to count them, we add up the lengths of each length.

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Example: Counting PINs without repeated

symbols

How many PINs of length 4 (with repetition) **do not** contain a repeated digit?

Consider the following:

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- When there are **no repetitions** of digits, there are P(10, 4) = 5040 PINs
- When there are **repetitions** of digits, there are $10^4 = 10000$ PINs

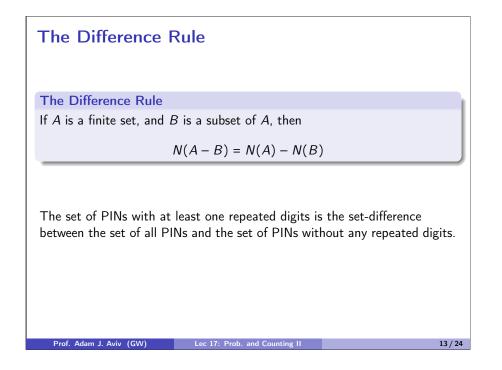
If we **subtract** the number of PINs that **do not** have repeated digits from the total number of PINs, we are left with just the PINs that have **at least** on repeated digit.

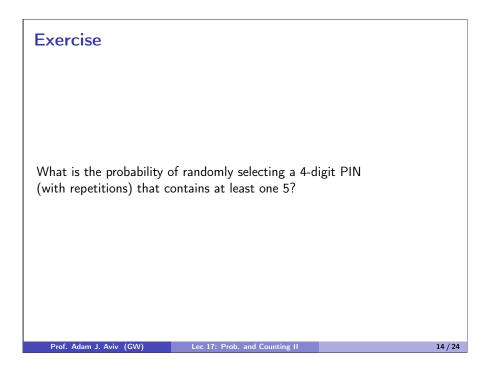
10000 - 5040 = 4060 PINs with at least one repeated digit

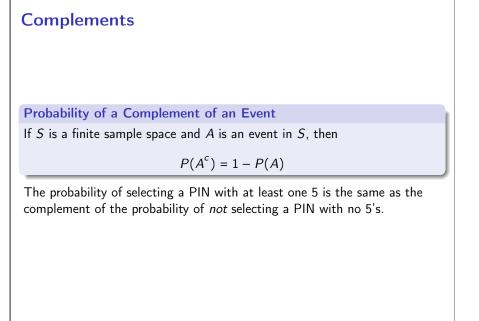
are divisible by 5?

Using the addition rule, how many three digit numbers 100-999

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Counting Unions (with overlaps)

How many numbers between 1 and 1000 are divisible by 3 or 5?

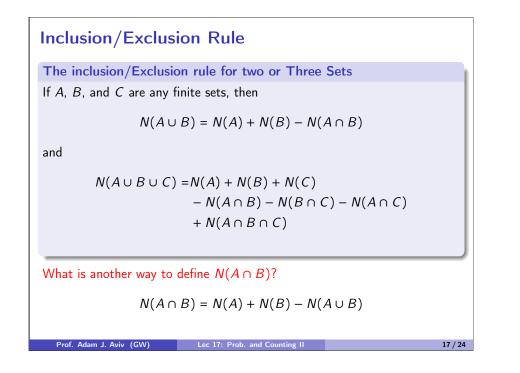
We can calculate the number of divisible numbers for both 3 and 5

- [1000/3] = 333
- [1000/5] = 200

But we can not take the sum of the two numbers to answer the question because some numbers are divisible by BOTH 3 and 5.

For a number to be visible by 3 and 5, it is divisible by 15, and $\lfloor 1000/15 \rfloor = 66$ would be counted twice. We can subtract that from the total.

Thus there are 333 + 200 - 66 = 467 numbers between 1 and 1000 that are divisible by 3 or 5.



Probability of General Unions and Intersections

Recall that a P(E), or the probability of E, is N(E)/N(S), that is, the number of ways an event can occur divided by the size of the sample space.

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What if we had two kinds events, A and B:

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- Probability of A or B is the same as P(A ∪ B)
 P(A ∪ B) = P(A) + P(B) P(A ∩ b)
- Probability of A and B is the same as P(A ∩ B)
 P(A ∩ B) = P(A) + P(B) P(A ∪ B)

Number of Subsets of a Given Size Consider the set, $\{A, B, C, D\}$, how many subsets of size 2 exist? • $\{A, B\}, \{A, C\}, \{A, D\}$ • $\{B, C\}, \{B, D\}$ • $\{C, D\}$ **Combinations and Poker Hands** This is less than the ways to permute 2 items from the set, in fact half the size AB, BA, CA, AC, AD, DA • BC, CB, BD, DB • CD, DC Calculate the number of subsets of size 3 exist? How does that compare to the number of permutations of size 3? Prof. Adam J. Aviv (GW) Lec 17: Prob. and Countin 19/24 Prof. Adam J. Aviv (GW) Lec 17: Prob. and Countir 20 / 24

Combination

Definition

Let *n* and *r* be non-negative integers with $r \le n$. An *r*-combination of a set of n elements is a subset of r of the n elements. We write a combination as

n r

read "*n* choose *r*". It is also denoted as C(n, r) and n C r.

A combination can be computed by dividing the number of permutations by the different arrangements, or r!.

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

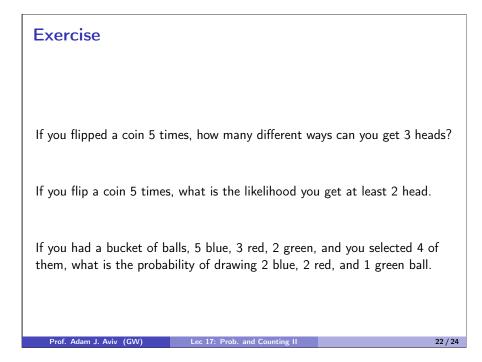
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Poker Hands

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Recall that a deck of cards has 4 suites, each with 13 cards: A, 2, 3, ..., 10, J, Q, K. A poker hand is a selection of 5 cards selected at random. How big is the sample space of poker hands? • $N(\text{poker-hands}) = \binom{52}{5} = 2598960$ How many poker hands contain at least a pair? Two cards that are the same? • P(at least one pair) = 1 - P(no-pair) = 1 - N(no-pair)/N(poker-hands)Choose 5 cards of differnt rank (13) 5 = 1 317 888 • N(no-pair) =• P(at least one pair) = 1 - 1317888/2598960 = 1 - 0.507 = 0.493 = 49.3%Prof. Adam J. Aviv (GW) Lec 17: Prob. and Counting II

