Lec 16:
Prob. and Counting I

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GW

CSCI 1311 Discrete Structures I
Spring 2023

Flipping Coins

When you flip a coin, there are two outcomes: Heads or Tails.

Suppose we toss two coins, what is the probability of getting *at least one* heads?

When tossing two coins, we can have either two heads, two tails, or one head and one tails. We'd expect then two of the three outcomes provides at least one head. **But that's incorrect.**

At least one heads . . .

We reached two-of-three by incorrectly counting the number of outcomes. Instead consider each coin flip indecently; we flip coin A and then coin B.

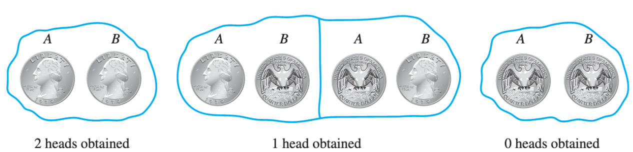


Figure 9.1.2 Equally Likely Outcomes from Tossing Two Balanced Coins

Instead, we see that there are in fact four possible outcomes. Of those four, three have at least one head, or three-out-of-four is the likelihood of flipping two coins and obtaining at least one head.

Random Process

Definition

A process is random when one outcome from a set of outcomes is sure to occur, but it is impossible to predict with certainty which outcome that will be.

For example, flipping a coin is described as random. It has two outcomes. One of those two outcomes, either heads or tails, are sure to occur on each coin flip, but we cannot not with certainty which of the outcomes it will be.

Sample Space and Random Event

Definition

A sample space is the set of all possible outcomes of a random process or experiment. an event is a subset of a sample space

The sample space of *a coin flip* is two $\{H, T\}$. An event, or random event, is either heads or tails.

The sample space of *two coin flips* is four $\{HH, HT, TH, TT\}$. An event would be one of those outcomes.

Equally Likely Probability Formula

Equally Likely Probability Formula

If S is a finite sample space in which all out comes are equally likely and E is an event in S, then the probability of E, denoted $P(E)$ is

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{the total number of outcomes in } S}$$

We can use $N(A)$ to denote the number of elements in a finite set A. So,

$$\boxed{P(E) = \frac{N(E)}{N(S)}}$$

Example: Playing Cards

A deck of cards has 52 cards in four suites, two black and two red

- ♣ : clubs
- ♦ : diamonds
- ♠ : spades
- ♥ : hearts

In each suite, there are 13 cards from A,2,3,4,5,6,7,8,9,J,Q,K, where A is an "Ace", J is a "jack", Q is a "queen" and K is a "king".

Imagine you a drawing a card from a sufficiently shuffled deck such that it is a random process.

Exercises

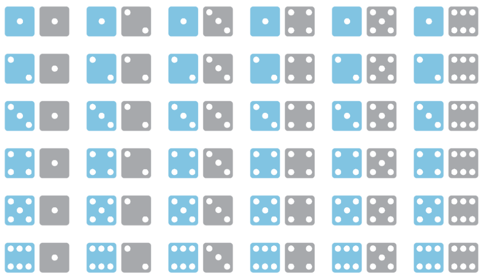
What is the sample spaces of outcomes for drawing a single card?

What are the events for which the card being drawn is a red suited card?

What is the probability that the chosen card is a Red Jack?

Rolling Dice

Like coins, rolling a die is a random events with 6 possible outcomes, labeled 1-to-6. If we roll two dice, there are 36 possible outcomes in our sample space.



And for simplicity, when discussing dice, we will simplify by writing two numbers for the dice roll, such as 24 would mean rolling a 2 and a 4.

Exercises

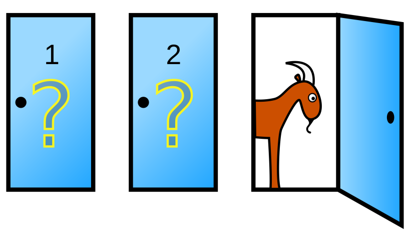
What is the size of the event space of rolling a total of two dice that adds up to 7?

What is the likelihood of rolling at least one 2 between the two dice?

What is the likelihood of rolling a total less than or equal to 3?

The "Monty Hall" problem

Monty Hall was a game show host of "Let's Make a Deal"



Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice? [1]

[1] vos Savant, Marilyn (9 September 1990). "Ask Marilyn". Parade Magazine: 16. Archived from the original on 21 January 2013.

Solving the Monty Hall Problem

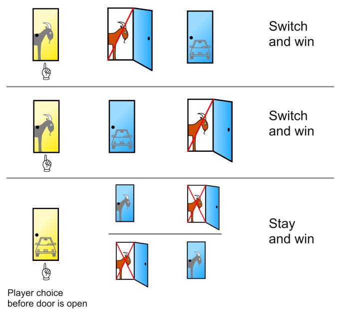
Amazingly, the likelihood of winning the car when you switch doors is 2/3, vs only 1/3 if you keep your original choice. How can this be?

The key to the analysis is to observe that there are three possible states after the first choice. After the host reveals a goat, in two of the three possible states, switching will lead to the car.

Analyzing the Monty Hall Problem

Consider that there are two goats, the first and second goat, and one car.

- If your first choice was the car, then the host can open either of the remaining doors, which contain goats. You swap away from the car to a goat. You loose.
- If your first choice was the first goat, the host is forced to show you the location of the second goat when revealing a door. You swap from the first goat to the car. You win!
- If your first choice was the second goat, the host is forced to show you the location of the first goat when revealing a door. You swap from the second goat to the car. You win!



In two of the three states, swapping doors leads to a win. The best strategy is to change to the other door.

Possibility Trees

Another way to think of modeling the Monty Hall problem (and other problems) is based on a Possibility Tree

```
                   First        After
                   Choice       Swapping

                ,-- (goat 1) -- (car)
               /
  Start (*)--- (goat 2) -- (car)
               \
                \.- (car) ----- (goat)
                        .
```

The tree branches on each choice from a root (start). The leaves (final states) shows all possibilities. We can see that two of the three possible states leads to a car, while only one leads to the goat.

Example: Two in a row of best of 5

Consider a tournament where two teams play repeatedly until one wins two in a row or a total of three games. We can model the possible outcomes of the tournament using a possibility tree.

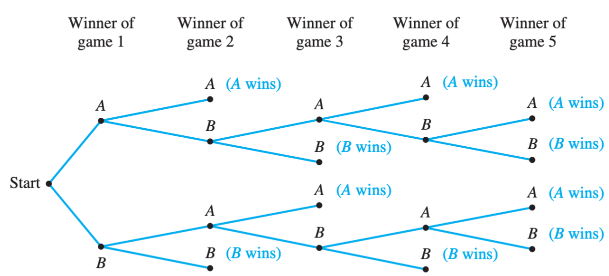


Figure 9.2.1 The Outcomes of a Tournament

What is the probability that 5 games are needed?

Multiple Events with Different Selection States

Consider a scenario where we want to flip a coin, and then roll a dice.
How many different outcomes could we have?

```
            Coin      Dice Roll
            Flip
                      1  2 3
                      | / /
                      |/ /
          .- (Heads)---*-<
         /            |\ \
        /             | \ \
       /              6  5 4
start (*)
       \              1  2 3
        \             | / /
         \            |/ /
          '-(Tails)---*-<
                      |\ \
                      | \ \
                      6  5 4

                 .
```

There are 2 possible outcomes after the coin flip, and there are 6 possible outcomes from the dice roll. Counting the total end states, there are 12 total, or $6 \cdot 2$.

Multiplication Rule

The Multiplication Rule

If an operation consists of k steps (each independently performed)

the first step can performed in n_1 ways
the second step can performed in n_2 ways
$\vdots$
the k-th step can performed in n_k ways

then the entire operation can be performed in $n_1 \cdot n_2 \cdot n_3 \cdots n_k$ ways.

The coin flip can occur in 2 ways, and the dice roll can occur in 6 ways. The entire operation of flipping a coin and rolling a dice can occur in $2 \cdot 6 = 12$ ways.

Exercise: PINs and Passwords

A PIN consists of a sequence of digits.

- How many 4 digit PINs exist?
- How many 6 digit PINs exist?
- How many 4 digit PINs exist where you cannot repeat digits?

Consider a password that consists of one capital letter followed by 4 digits, and then two capital letters.

- How many possible passwords exist?
- Does the order of the letters and numbers change that calculation?

Consider a password that can consists of exactly 8 upper, lower case letters or numbers.

- How many possible passwords exist?

Independent/Dependent Events

A naive application of the multiplication will work in many situations, as long as the events are independent. That is, the number of ways to perform an operation in the k-th step does not depending on an earlier step.

However, if events are dependent we may need to consider prior events in determine the number of ways the next step can be taken.

Example

Three officers–a president, a treasurer, and a secretary–are to be chosen from among four people: Ann, Bob, Cyd, and Dan. Suppose that, for various reasons, Ann cannot be president and either Cyd or Dan must be secretary. How many ways can the officers be chosen?

Example: Choosing Officers

Incorrect Analysis: There are three choices for president (all but Ann). three choices for treasurer (all but the president), and two choices for secretary (Cyd or Dan); $3 \cdot 3 \cdot 2 = 18$.

But choosing a secretary depends on who was selected as president. If Cyd was selected as president, then only Dan can be secretary.

Correct Analysis: We can use a possibility tree to represent the correct multiplication of possibilities. There only 8 ways to select officers.

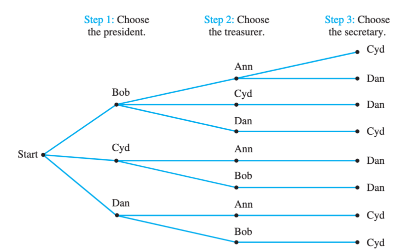


Figure 9.2.3

We can reorder the choices, though, so we can directly apply the multiplication rule. Consider choosing the secretary, president, and then treasurer: are there any dependencies?

Exercise

Suppose you are choosing passwords of length 4 where each item can either be a digit $[0, 9]$ or a capital letter $A, B, C, \ldots, Z$.

A password must begin with a letter and end with a number. How many possible passwords exist?

A password must begin with a letter, end with a number, and no symbol can be used more than once. How many possible passwords exist?

A password must begin with a letter [A-G], end with a number [0-2], and no symbol can be used more than once. How many possible passwords exist?