Lec 14: Relations I

Prof. Adam J. Aviv

GW

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Relations between sets

Relations, like functions, are a way to describe how elements in one set relate to elements in an another set.

Example

Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3\}$. Then $x \in A$ is related to $y \in B$ if, and only if, $x < y$. Symbolically

$$x \, R \, y \iff x < y$$

Where $x \, R \, y$ is read "x related to y."

Enumerate all pairs $(x, y) \in A \times B$ that are *in* the relation?

Relations as sets (and subsets)

Moreover, we can define the relation R as a set itself. Again, if R is a relation between two sets A and B and $x \in A$ and $y \in B$, then R can be defined such that:

$$x \, R \, y \iff (x, y) \in R$$

Definition

Let A and B be sets. Then a relation R from A to B is a subset of $A \times B$. Given pair $(x, y) \in A \times B$, x is related to y by R, written $x \, R \, y$, if, and only if, $(x, y) \in R$.

The set A is the domain of R, and set B is the co-domain of R.

Reflexive, Symmetric and Transitive

Let R be a relation on a set A:

- R is **reflexive**, if and only if,
  $\boxed{\forall x \in A, x \, R \, x}$
  - All elements in A are related to themselves.
- R is **symmetric** if, and only if,
  $\boxed{\forall x, y \in A, x \, R \, y \implies y \, R \, x}$
  - For any x, y in A, if x is related to y, then y is related to x
- R is **transitive**, if, and only if,
  $\boxed{\forall x, y, z \in A, x \, R \, y \wedge y \, R \, z \implies x \, R \, z}$
  - For any x, y, z in A, if x is related to y and y is related to z, then x is related to z.

Exercise

How would you define (formally) the inverse of

R is ***not*** **reflexive** if, and only if, ...

R is ***not*** **symmetric** if, and only if, ...

R is ***not*** **transitive** if, and only if, ...

Exercise

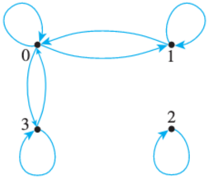
Let $A = \{0, 1, 2, 3\}$ and define the relation R as

$$R = \{(0,0), (0,1), (0,3), (1,0), (1,1), (2,2), (3,0), (3,3)\}$$

Is R reflexive? symmetric? transitive?

Direct graph depiction of relations

$$R = \{(0,0),(0,1),(0,3),(1,0),(1,1),(2,2),(3,0),(3,3)\}$$



- Reflexive requires self loops for all elements in A
- Symmetry requires loops to exists between any related elements
- Transitivity requires closed connections between co-related terms. So there should be an arrow between 3-to-1 and from 1-to-3.

Equivalence Relations

Definition

Let A be a set and a R a relation on A. R is an equivalence relation if, and only if, R is reflexive, symmetric, and transitive.

Note that a relation is not an equivalence relation if there exists at least one counterexample of reflexivity, symmetry, or transitivity.

Example equivalence relation: $=$

Define R as a relation over the reals such that

$\forall x, y \in \mathbb{R}$

$$\boxed{x \, R \, y \iff x = y}$$

Two items are in the relation if they are equal: is this an equivalence class?

That is, is R reflexive, symmetric and transitive?

$=$ as an equivalence relation

Proof.

For R **to be reflexive**, we must show that $\forall x \in \mathbb{R}, x\,R\,x$, or equivalently show that $x = x$, which is true for equality, so R is reflexive.

For R **to be symmetric**, we must show that $\forall x, y \in \mathbb{R}, x\,R\,y \implies y\,R\,x$, or equivalently, $x = y \implies y = x$, which is certainly true for equality of the reals, so R is symmetric.

For R **to be transitive**, we must show that $\forall x, y, z \in \mathbb{R}, x\,R\,y \wedge y\,R\,z \implies x\,R\,z$, or equivalently, that if $x = y$ and $y = z$, then $x = z$, which is also true for equality of the reals, so R is transitive.

Thus R is a equivalence relation. □

Less than relation as equivalence?

Define the relation R over the reals as

$\forall x, y \in \mathbb{R}$

$$x \, R \, y \iff x < y$$

Is the less than relation R an equivalent relation?

Exercises

Are the following equivalence relations? Try to prove your result, and if they are not, which property is violated.

Let R be the relation on $\mathbb{Z}^+$ such that

$$(\forall n, m \in \mathbb{Z}^+)(n\,R\,m \iff n|m)$$

Let S be the relation on $\mathbb{R}$ such that

$$(\forall x, y \in \mathbb{R})(x\,S\,y \iff xy \geq 0)$$

Antisymmetry

Definition

Let R be a relation on set A. R is atisymmetric if, and only if, forall a and b in A, if $a\,R\,b$ and $b\,R\,a$ then $a = b$

More formally

$$R \textbf{ is antisymmetric} \iff (\forall a, b \in A)[(a\,R\,b \;\wedge\; b\,R\,a) \implies a = b]$$

Contrapositive defines **not antisymmetric**

$$R \textbf{ is not antisymmetric} \iff (\exists a, b \in A)(a\,R\,b \;\wedge\; b\,R\,a \wedge a \neq b)$$

Recall that $\neg(p \to q) \equiv p \wedge \neg q$

Example antisymetric relation: divides

Let R be a relation over $\mathbb{Z}^+$, and $a\,R\,b$ if, and only if, $a \mid b$. R is antisymetric.

Proof.

Suppose that $a\,R\,b$ and $b\,R\,a$, we must show that $a = b$. By definition of the relation $a\,R\,b$ and $b\,R\,a$ implies that

$$\begin{aligned} a\,R\,b &\implies a \mid b \implies a = br \text{ for some positive integer r} \\ b\,R\,a &\implies b \mid a \implies b = as \text{ for some positive integer s} \end{aligned}$$

Substituting for b in the first formula we have

$$\begin{aligned} a &= br \\ a &= (as)r \\ 1 &= sr \end{aligned}$$

Since s and r are positive integers, they must be 1. Substituting for r we have $a = 1b$ and thus d $b = a$. □

Is the relation R antisymmetric over the positive and negative integers?

Partial order relation

Definition

A relation R over a set A is a partial order relation if, and only if, R is reflexive, antisymmetric, and transitive.

Antisemitic ensures that we can create some chain of ordering over elements. For example, for the divides relation $a\,R\,b$ if, and only if, $a \mid b$, we can say that $a \preceq b$ if $a\,R\,b$, leading to various ordered chains factors, all ending at 1.

$$120 \succeq 10 \succeq 5 \succeq 1$$
$$100 \succeq 20 \succeq 4 \succeq 2 \succeq 1$$

We use the $\preceq$ and $\succeq$ to indicate "less than or equal" or "greater than or equal" under the relation as not to be confused with the canonical less/greater than $(\leq, \geq)$.

Exercise

Let $\mathcal{A}$ be a collection of sets. Show that $\subseteq$ is a partial ordering over $\mathcal{A}$, that is for sets a and b in $\mathcal{A}$, $a\,R\,b$ if, and only if, $a \subseteq b$.

Total Ordering

In partial ordering, it is possible to find two elements a and b that are not related. For example, in the divides relation, $a \nmid b$ would not be in the relation, and thus a and b could not be ordered.

Definition

A partial order relation R on set A is a total order relation if for any two elements a and b either $a \, R \, b$ or $b \, R \, a$.

Example: The less than relation $a \leq b$ is a total order relation.