Lec 12:
Functions I

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Functions

You've been exposed to many different kinds of functions:

- Functions in programming contain code segments with clear input and output.
- Functions in continuous forms, i.e, continuous functions in the Cartesian plane. Such as $f(x) = x^2 + 2x + 1$.

Mathematical Definition of Functions

Definition

A function is a relation between two sets, written as $f : X \to Y$, where X is the set of inputs (or the domain) and Y is the set of possible outputs Y (the co-domain), that satisfies two properties:

- every element in X is related to some element in Y
- no element in X is related to more than one element in Y

This arrow diagram does define a function because

1. Every element of X has an arrow coming out of it.
2. No element of X has two arrows coming out of it that point to two different elements of Y.

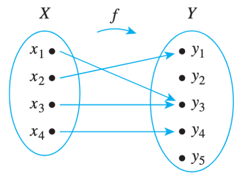

Figure 7.1.1

Range/Image and Inverse-Image/Pre-Image of f

Range (or Image of X under f) is the set of $y \in Y$ such that there exists an $x \in X$ that f maps x to y (or $x \to^f y$).

$$\textbf{image of } y = \{y \in Y \mid \exists x \in X \text{ s.t. } f(x) = y\}$$

Inverse image of $y \in Y$ (or preimage) is the set of $x \in X$ for which $x \to^f y$

$$\textbf{preimage of } y = \{x \in X \mid f(x) = y\}$$

Exercise

Define the function $g : \mathcal{P}(\{a, b, c\}) \to Z^{+}$ where $g(X) = |X|$.

- Draw an arrow diagram for g.
- What is the image of g?
- What is the preimage of 2?

Thinking of a function as a set (or relation)

Another way to think about functions is as a *set of input and output pairs*. For example, in the function $f : X \to Y$, we could also define:

$$f = \{(x, y) \mid f(x) = y\}$$

Which provides the following biconditional,

$$(x, y) \in f \iff (\exists x \in X)(\exists y \in y)(y = f(x))$$

Additionally, $f \subseteq X \times Y$

Equivalence of Functions

Theorem (Test for function equivalence)

$F : X \to Y$ and $G : X \to Y$ are (set) equivalent $F = G$ if, and only if, $\forall x \in X, F(x) = G(x)$

Proof.

Note that $F \subseteq X \times Y$ and $G \subseteq X \times Y$.

- $\Leftarrow$: Suppose $\forall x \in X, F(x) = G(x)$, show that $F = G$

  If $(x, y) \in F$, then $y = F(x)$, and also if $y = G(x)$ then $(x, y) \in G$. F and G contain the same elements, and are thus set equivalent.

- $\Rightarrow$: Suppose that $F = G$, show that $\forall x \in X, F(x) = G(x)$

  If F and G contain the same elements, then for all $(x, y) \in F$, $F(x) = y$ and also $(x, y) \in G$, $G(x) = y$. So $\forall x \in X$, $G(x) = F(x)$. □

Exercise

Let $F : \{0, 1, 2, 3, 4\} \to \{0, 1, 2\}$ and $G : \{0, 1, 2, 3, 4\} \to \{0, 1, 2\}$, if

$$F(x) = (x^2 + x + 1) \mod 3 \qquad \text{and} \qquad G(x) = (x + 2)^2 \mod 3$$

show that $F = G$

Hint: this would be like a truth table with more inputs

One-to-one functions

Definition

A function $f : X \to Y$ is one-to-one (or injective) if, and only if, for all elements x_1 and x_2 in X,

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

Or equivalently by the contrapositive

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

We can write this symbolically as

$$f : X \to Y \text{ is one-to-one} \iff (\forall x_1, x_2 \in X)(f(x_1) = f(x_2) \implies x_1 = x_2)$$

What is definition of not one-to-one?

What is the contrapositive?

$$f : X \to Y \text{ is one-to-one} \iff (\forall x_1, x_2 \in X)(f(x_1) = f(x_2) \implies x_1 = x_2)$$

Finding the contrapositive of a biconditional:

$$\begin{aligned} p \leftrightarrow q &\equiv (p \to q) \wedge (p \leftarrow q) \\ &\equiv (\neg q \to \neg p) \wedge (\neg q \leftarrow \neg p) \\ &\equiv \neg q \leftrightarrow \neg p \\ &\equiv \neg p \leftrightarrow \neg q \end{aligned}$$

Applying that to our statement, we find that:

$$\begin{aligned} &\boxed{f : X \to Y \text{ is } \textbf{not} \text{ one-to-one}} \\ &\iff \neg[(\forall x_1, x_2 \in X)(f(x_1) = f(x_2) \implies x_1 = x_2)] \\ &\iff (\exists x_1, x_2 \in X)\neg[(f(x_1) = f(x_2) \implies x_1 = x_2)] \\ &\iff \boxed{(\exists x_1, x_2 \in X)(f(x_1) = f(x_2) \wedge x_1 \neq x_2)} \end{aligned}$$

...when there exists x_1 and x_2, s.t. $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Exercise

Consider $A = \{1, 2, 3\}$ and $B = \{w, x, y, z\}$ and possible functions below.

Are these functions well-defined and if so, are they one-to-one?

$f = \{(1, w), (2, x), (3, x), (3, z)\}$

$g = \{(1, w), (2, x), (3, x)\}$

$h = \{(1, w), (2, x), (3, z)\}$

How many one-to-one functions exist between A and B?

Proving a function is one-to-one

If we want to prove a function is one to one, we follow the definition

$$f : X \to Y \text{ is one-to-one} \iff (\forall x_1, x_2 \in X)(f(x_1) = f(x_2) \implies x_1 = x_2)$$

We must show that if $f(x_1) = f(x_2)$, then $x_1 = x_2$ for all x_1 and x_2.

So we get to assume that $f(x_1) = f(x_2)$, and we must show that $x_1 = x_2$.

Example proving one-to-one

Theorem

$f : \mathbb{R} \to \mathbb{R}, f(x) = 4x - 1$ is one-to-one

Proof.

We must show that $x_1 = x_2$ for all $x_1, x_2 \in \mathbb{R}$ whenever $f(x_1) = f(x_2)$. Prove this directly, starting with $f(x_1) = f(x_2)$

$$\begin{aligned} f(x_1) &= f(x_2) & \\ 4x_1 - 1 &= 4x_2 - 1 & \text{expand } f(x) \\ 4x_1 &= 4x_2 & \text{add 1 to both sides} \\ x_1 &= x_2 & \text{divide both sides by 4} \end{aligned}$$

□

Exercise

Prove that the following are one-to-one, or provide a counter example.

$f : \mathbb{Z} \to \mathbb{Z}, f(x) = x^2$

$g : \mathbb{Z} \to \mathbb{Z}, g(x) = x^3$

$h : \mathbb{R}^+ \to \mathbb{Z}^+, h(x) = \lfloor x \rfloor$

$\lfloor \cdot \rfloor$ is the floor function, it lowers any real number to the smallest integer value less than that number. For example, $\lfloor 7.75 \rfloor = 7$ and $\lfloor -3.2 \rfloor = -4$

Onto Functions

Definition

A function $f : X \to Y$ is onto (or surjective) if, and only if, given any element $y \in Y$, it is possible to find an element $x \in X$ such that $f(x) = y$.

We can write this symbolically as

$$f : X \to Y \text{ is onto} \iff (\forall y \in Y)(\exists x \in X)(f(x) = y)$$

The contrapositive reveals a definition for not onto

$$f : X \to Y \text{ is } \textbf{not} \text{ onto} \iff (\exists y \in Y)(\forall x \in X)(f(x) \neq y)$$

Proving something is onto (or not onto)

If we wanted to show some function is onto, we can follow the definition

$$f : X \to Y \text{ is onto} \iff (\forall y \in Y)(\exists x \in X)(f(x) = y)$$

We would need to provide an example (or formula for) x such that $f(x) = y$ for all y.

If we wanted to show that something is **not** onto, again following the definition.

$$f : X \to Y \text{ is } \textbf{not} \text{ onto} \iff (\exists y \in Y)(\forall x \in X)(f(x) \neq y)$$

We can find a counter example y for which there is no corresponding x where $f(x) = y$

Example proof

Theorem

$f : \mathbb{R} \to \mathbb{R}, f(x) = 4x - 1$ is onto

Proof.

Assume that $y \in \mathbb{R}$, we must find a $x \in \mathbb{R}$ where $f(x) = y$. Let's assume x exists for every y, then

$$y = 4x - 1$$
$$\frac{y+1}{4} = x$$

The formula for $x = \frac{y+1}{4}$ always provides real number. If we apply it to $f(x)$

$$f(x) = f\left(\frac{y+1}{4}\right) = 4\left(\frac{y+1}{4}\right) - 1$$
$$= y + 1 - 1 = y$$

You always get y, thus this formula for an x can produce any y. □

Exercise

Prove that the following functions are onto, or provide a counter example.

$f : \mathbb{Z} \to \mathbb{Z}, f(x) = 4(x - 1)$

$g : \mathbb{R} \to \mathbb{R}, g(x) = x^2$

One-to-one Correspondence Functions

Definition

A one-to-one correspondence (or bijection) from set X to Y is a function $f : X \to Y$ that is both one-to-one and onto.

Example

The function $f : \mathbb{R} \to \mathbb{R}, f(x) = 4x - 1$ is one-to-one and onto, and is thus a bijection of the real numbers (or a one-to-one correspondence function)

Example of one-to-one Correspondence Functions

Prove that

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}, f(x, y) = (x + y, x - y)$$

is a one-to-one correspondence function.

We must prove that the function is both one-to-one and onto.

Proof of one-to-one

Proof of one-to-one.

We must show that if $f(x_1, y_1) = f(x_2, y_2)$ then $x_1 = x_2$ and $y_1 = y_2$.

$$f(x_1, y_1) = f(x_2, y_2)$$
$$(x_1 + y_1, x_1 - y_1) = (x_2 + y_2, x_2 - y_2)$$

For two pairs to be equal, the first and second element must be equal.

$$x_1 + y_1 = x_2 + y_2$$
$$x_1 - y_1 = x_2 - y_2$$

Adding the equations gives $2x_1 = 2x_2$, or $x_1 = x_2$. After substitution.

$$x_1 + y_1 = x_2 + y_2$$
$$x_1 + y_1 = x_1 + y_2$$
$$y_1 = y_2$$

So $x_1 = x_2$ and $y_1 = y_2$, when $f(x_1, y_1) = f(x_2, y_2)$ and f is one-to-one. □

Proof of onto

Proof of onto.

Assume that (u, v) is in the co-domain of f, we must show that there exists input (r, s) such that $f(r, s) = (u, v)$. If we let $r = \frac{u+v}{2}$ and $s = \frac{u-v}{2}$ then.

$$\begin{aligned} f\left(\frac{u+v}{2}, \frac{u-v}{2}\right) &= \left(\frac{u+v}{2} + \frac{u-v}{2}, \frac{u+v}{2} - \frac{u-v}{2}\right) \\ &= \left(\frac{u+v+u-v}{2}, \frac{u+v-u+v}{2}\right) \\ &= \left(\frac{2u}{2}, \frac{2v}{2}\right) \\ &= (u, v) \end{aligned}$$

So for any (u, v) in the co-domain, we can find an input (r, s), and thus f is onto. □

Inverse Function

If a function $f : X \to Y$ is a one-to-one correspondence, then there must exist an inverses function $f^{-1} : Y \to X$

$$f^{-1}(y) = x \iff y = f(x)$$

Why would a function that is just onto or one-to-one, but not both, not have an inverse function?

What is the inverse function for $f : \mathbb{R} \to \mathbb{R}, f(x) = 4x - 1$?