Lec 10:
Recurrence Relations

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GW

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Recurrence Relations

Definition

A recurrence relation for a sequence $a_0, a_1, a_2, \ldots$ is a formula that relates each term a_k to i-number of its predecessors $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$, where $k - i \geq 0$.

The initial condition for a recurrence relation specifies the values of $a_0, a_1, a_2, \ldots, a_{i-1}$ needed to start evaluation.

Example

Recurrence relation for the sum of the positive integers:

$$n_1 = 1 \qquad n_k = n_{k-1} + 1$$

Any positive integer is defined as one more than the previous positive integer. For example n_4 is

$$n_4 = n_3 + 1 = (n_2 + 1) + 1 = ((n_1 + 1) + 1) + 1 = ((1 + 1) + 1) + 1 = 4$$

Exercises

Find the 4th term (a_4 and c_4) of the following recurrence relations

$$a_k = 2a_{k-1} + 5$$
$$a_1 = 4$$

$$c_k = c_{k-1} + k \cdot c_{k-2} + 1$$
$$c_0 = 1 \quad c_1 = 2$$

Tower of Hanoi

On the steps of the altar in the temple of Benares, for many, many years Brahminshave been moving a tower of 64 golden disks from one pole to another; one by one, neverplacing a larger on top of a smaller. When all the disks have been transferred the Towerand the Brahmins will fall, and it will be the end of the world.

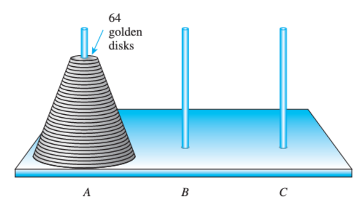
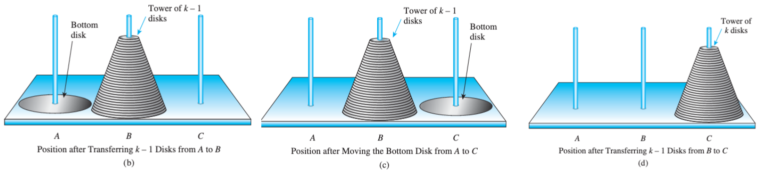


Figure 5.6.2

When will the world end? — Assume moving one disk takes 1 second.

Thinking about the problem recursively

Let M_k be the number of disk-moves to get k disks from one pole to another.



The cost to move k disks (M_k) disks from poll A to poll C equals:

- Cost of moving $k-1$ disks from A to B (cost is M_{k-1})
- Cost of moving Bottom Disk from A to C (cost is 1)
- Cost of moving $k-1$ disks from B to C (cost is M_{k-1})

Recurrence relation for Tower of Hanoi (1)

$$M_k = 2\underbrace{M_{k-1}} + 1$$

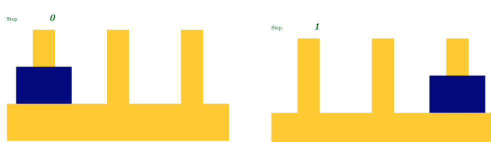
$$M_k = 2(\overbrace{\underbrace{2M_{k-2}}+1}) + 1$$

$$M_k = 2(2(\overbrace{\underbrace{2M_{k-3}}+1}) + 1) + 1$$

$$M_k = 2(2(2(\ldots 2(\overbrace{2M_0 + 1}) + 1\ldots) + 1) + 1) + 1$$

The base case: M_1

With one disk, move it directly from A to C.



$$M_1 = 1$$

Recurrence relation for Tower of Hanoi (2)

The number of moves M_k for k disks is defined as:

$$M_k = 2M_{k-1} + 1$$
$$M_1 = 1$$

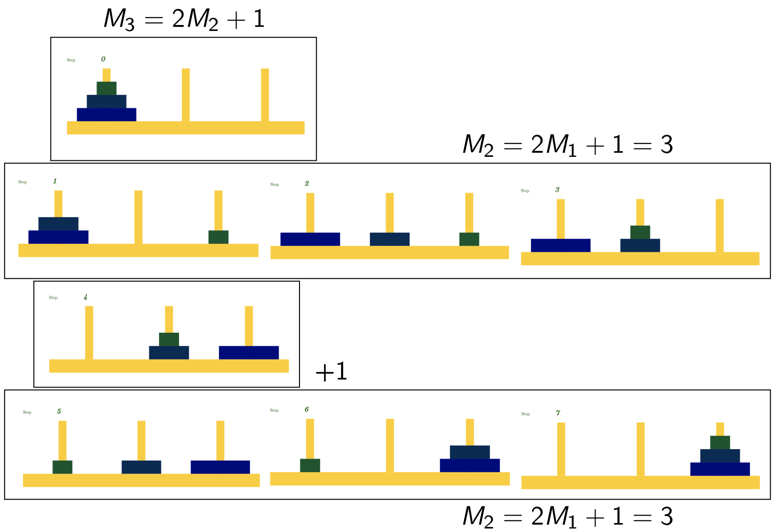
Example

Number of moves for a tower with 3 disks?:

$$M_3 = 2M_2 + 1 = 2(2M_1 + 1) + 1 = 2(2(1) + 1) + 1 = 7$$

How many moves for a tower with 4 disks?

Three-disk Hanoi Tower visualized



When does the world end?

Let's assume that each disk can be moved in 1-second, we can write a short recursive program to calculate M_k:

```
#M_n
def hanoi(n):
  #M_1 = 1
  if n == 1: return 1

  #M_k = 2*M_{k-1}+1
  else: return 2*hanoi(n-1) + 1
```

$$\texttt{hanoi(64)} = 18446744073709551615 \text{ (seconds)}$$
$$\cong 584.5 \times 10^9 \text{ (years)}$$

Solving a recurrence relation

Consider the following recurrence relation:

$$a_k = 2 + a_{k-1}$$
$$a_0 = 1$$

What if we wanted to find a formula for a_k without recurrences?

Routine for solving a recurrence relation

1. Write down the recurrence relation.
2. Expand the recurrence relation for several lines. Note how many times you do so, and don't reduce too much.
3. Determine what the i'th line would be.
4. Solve for what value of i would reach the base case
5. In your equation from (3), replace i with the solution from (4).

(1-3)Expand for several lines, solve for i

$$\begin{aligned} a_k &= 2 + a_{k-1} && \text{step 1} \\ &= 2 + 2 + a_{k-2} && \text{step 2} \\ &= 2 + 2 + 2 + a_{k-3} && \text{step 3} \\ &= 2 \cdot 3 + a_{k-3} \\ &\vdots \\ &= 2(i) + a_{k-i} && \text{step i} \end{aligned}$$

(4-5) Determine base case, and replace

On the i-th step the formula is:

$$a_k = 2(i-1) + a_{k-(i-1)}$$

Base case is $k = 0$, so when does $k - i = 1$? When $i = k$.

$$\begin{aligned} a_k &= 2i + a_{k-i} \\ &= 2k + a_{k-k} \\ &= 2k + a_0 \\ &= 2k + 1 \end{aligned}$$

Exercise

Solve the following recurrence relations and check your formulat out for three iterations, e.g., up to t_3 and s_3

$$\begin{aligned} t_k &= t_{k-1} + 9 \\ t_0 &= 11 \end{aligned}$$

$$\begin{aligned} s_n &= n + s_{n-1} \\ s_0 &= 0 \end{aligned}$$

Exercise (cont)

Prove your solution to the recurrence relations, using induction.

If $t_k = t_{k-1} + 9$ and $t_0 = 11$, then $t_k = 11 + 9k$

If $s_n = n + s_{n-1}$ and $s_0 = 0$, then $s_n = n(n+1)/2$

Solution for Recurrence of Tower of Hanoi (1)

Recall that the formula for the Tower of Hanoi is

$$M_k = 2M_k + 1 \quad M_1 = 1$$

$$\begin{aligned}
M_k &= 2M_{k-1} + 1 && \text{step 1}\\
&= 2(2M_{k-2} + 1) + 1 && \text{step 2}\\
&= 4M_{k-2} + 2 + 1 &&\\
&= 4(2M_{k-3} + 1) + 2 + 1 && \text{step 3}\\
&= 8M_{k-3} + 4 + 2 + 1 &&\\
&\ \ \vdots && \text{let } i = 3\\
&= 2^i M_{k-i} + 2^{i-1} + 2^{i-2} + \ldots 2^0 && \text{step i}\\
&= 2^i M_{k-i} + \sum_{j=0}^{i-1} 2^j &&
\end{aligned}$$

Solution for Tower of Hanoi (2)

The base case is when $k = 1$, so when $i = k - 1$, we reach a base case:

$$\begin{aligned}
&= 2^i M_{k-i} + \sum_{j=0}^{i-1} 2^j && i = k - 1\\
&= 2^{k-1} M_{k-(k-1)} + \sum_{j=0}^{(k-1)-1} 2^j &&\\
&= 2^{k-1} M_1 + \sum_{j=0}^{k-2} 2^j && M_1 = 1\\
&= \underbrace{2^{k-1} + \sum_{j=0}^{k-2} 2^j}_{\text{Same as summing to } k-1} = \underbrace{\sum_{j=0}^{k-1} 2^j}_{\text{Geometric Sum}} &&\\
&= \frac{1 - 2^k}{1 - 2} = 2^k - 1 &&
\end{aligned}$$

Proving the Tower of Hanoi

With a solution, we can prove it using induction:

Theorem

The k-th step of the recurrence relation $M_k = 2M_k + 1$, where $M_1 = 1$, for all $k > 1$, is $2^k - 1$

Proof.

By induction on k,

Base Case: k=2: $M_2 = 2(1) + 1 = 3 = 2^2 - 1$

Inductive step: Assume $M_k = 2^k - 1$ for all, show that $M_{k+1} = 2^{k+1} - 1$.

Note that, $M_{k+1} = 2M_k + 1$ and by the induction hypothesis, $M_k = 2^k - 1$. Substituting in, we have
$M_{k+1} = 2(2^k - 1) + 1 = 2^{k+1} - 2 + 1 = 2^{k+1} - 1$, showing our result. □

(extra) Exercises

Solve the following recurrence relations and prove the result using induction on k

$t_k = 2k + t_{k-1} \quad t_0 = 2$

$t_k = 3t_{k-1} + 2 \quad t_1 = 3$