Lec 08:
Induction I

Prof. Adam J. Aviv

GW

CSCI 1311 Discrete Structures I
Spring 2023

The sum of all numbers between 1-100

A folk tale about Gauss is that he was an annoying and disruptive elementary school student, so much so, that his teacher asked him to sum all the numbers between 1-100 to distract him to get some peace and quite in the classroom.

But Gauss found the answer within a few seconds, without having to sum up all the numbers up directly.

Carl Friedrich Gauss
1777-1855

Gauss' "trick"

$$\begin{array}{cccccccc}
 & 1\,+ & 2\,+ & 3\,+ & \ldots & +\,98 & +\,99 & +\,100 \\
+ & 100\,+ & 99\,+ & 98\,+ & \ldots & +\,3 & +\,2 & +\,1 \\
\hline
 & 101\,+ & 101\,+ & 101\,+ & \ldots & +\,101 & +\,101 & +\,101
\end{array}$$

$$\underbrace{\hspace{10em}}_{\text{100 items}}$$

$$2 \cdot \sum_{i=1}^{100} = 100 \cdot 101 = 10100$$

Dividing by 2 gives us the result

$$\sum_{i=1}^{100} = (100 \cdot 101)/2 = 5050$$

Generalizing Gauss' "trick"

What if we wanted to find a formula for the sum of the numbers 1-to-n?

$$\begin{array}{cccccccc}
 & 1\,+ & 2\,+ & 3\,+ & \ldots & +\,n-2 & +\,n-1 & +\,n \\
+ & n\,+ & n-1\,+ & n-2\,+ & \ldots & +\,3 & +\,2 & +\,1 \\
\hline
 & n+1\,+ & n+1\,+ & n+1\,+ & \ldots & +\,n+1 & +\,n+1 & +\,n+1
\end{array}$$

$$\underbrace{\hspace{10em}}_{n\text{ items}}$$

$$2 \cdot \sum_{i=1}^{n} = n(n+1)$$

$$\sum_{i=1}^{n} = \frac{n(n+1)}{2}$$

Proving this based just on the equation

What if I asked you to prove it based on the equation alone?

$$\forall n \in \mathbb{Z}^+, \sum_{1}^{n} = \frac{n(n+1)}{2}$$

We can test for specific values,

$$\sum_{i=1}^{3} i = 1 + 2 + 3 = 6 = 3(3-1)/2$$

$$\sum_{i=1}^{10} i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55 = 10(10+1)/2$$

As a universal quantifier, we can't check for all n.

Definition of (Weak) Induction

Definition (Method of Proof by Mathematical Induction)

To prove that $\forall n \geq a$, a property $P(n)$ is true, we must prove two cases:

- **Base Case:** $P(a)$ is true
- **Inductive Step:** $(\forall k \geq a)(P(k) \implies P(k+1))$ is true.

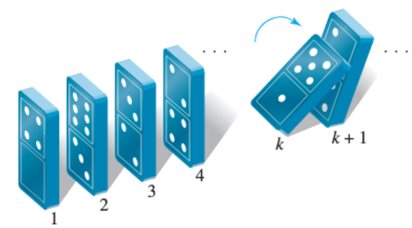

Figure 5.2.3 If the kth domino falls backward, it pushes the $(k+1)$st domino backward also.

The Inductive Hypothesis (IH)

Note the inductive step requires that you prove an implication:

$$P(k) \implies P(k+1)$$

That is, "if $P(k)$, then $P(k+1)$."

Because we are proving an implication, we can assume that the hypothesis of the implication, $P(k)$, is true in proving that $P(k+1)$ is true.

The assumption that $P(k)$ is true is referred to as the inductive hypothesis.

Proving Gauss' formula with induction (1)

Theorem

For all integers $n \geq 1$, $\sum_{i=1}^{n} i = n(n+1)/2$

Let $P(k)$ be the preposition that the sum of the integers 1-to-k is equivalent to the formula $k(k+1)/2$. We must show this to be true $\forall k \geq 1$

We must prove two cases.

Base case: $P(1)$

Inductive case: $(\forall k \geq 1)(\underbrace{P(k)}_{\textbf{Assume IH}} \implies \overbrace{P(k+1)}^{\textbf{Prove this}})$

Proving Gauss' formula with induction (2)

The base case is straightforward. This is clearly true at P(1).

Proof of Base Case: $P(1) \equiv \sum_{i=1}^{1} i = 1(1+1)/2$.

This is directly true because $\sum_{i=1}^{1} = 1$ and $1(1+1)/2 = 1$. □

Describing the inductive step

Assume the inductive hypothesis (IH)

$$P(k) \equiv \overbrace{\left[\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\right]}^{\text{Assume this equality is true!}}$$

Can we show?

$$P(k+1) \equiv \underbrace{\left[\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}\right]}_{\text{Can we show this equality is true?}}$$

Inductive step (1)

Proof of Inductive Step: $P(k) \implies P(k+1)$

We can rewrite the sum of the integers formula 1-to-$(n+1)$ with two sums.

$$\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1)$$

By the inductive hypothesis, $\sum_{i=1}^{k} i = k(k+1)/2$

$$\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1)$$

Inductive step (2)

Proof of Inductive Step: $P(k) \implies P(k+1)$ **(cont.)**

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \frac{k(k+1)}{2} + (k+1)\\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}\\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$

Which shows our result. □

Exercise

Proof the following using induction.

$\sum_{i=1}^{n}(6i-2) = n(3n+1)$

For all integers $n \geq 1$, $2^n - 1$ is odd.

For all integers $n \geq 1$, $5^n - 1$ is divisible by 4.

Arithmetic and Geometric Sequence

The sum of the first n integers is also called the arithmetic sum and comes up often in computer science.

$$\sum_{i=1}^{n} i \qquad \textbf{Arithmetic Sum}$$

The geometric sum is another import summation in computer science that relates to exponentiation of ratios.

$$\sum_{i=0}^{n-1} r^i \qquad \textbf{Geometric Sum}$$

Where $r \in \mathbb{Q}$ (set of rationals).

Deriving the Geometric sequence (1)

Consider the sequence

$$\sum_{i=0}^{n-1} = r^0 + r^1 + r^2 + \ldots + r^{n-3} + r^{n-2} + r^{n-1}$$

If we multiply the sequence by $(1-r)$, it subtracts the sequence from the sequence multiplied by r

$$(1-r)(r^0 + r^1 + r^2 + \ldots + r^{n-3} + r^{n-2} + r^{n-1}) =$$

$$\begin{array}{lccccccccc} & r^0 + & r^1 + & r^2 + & \ldots & + r^{n-3} & + r^{n-2} & + r^{n-1} & \\ - & & r^1 + & r^2 + & \ldots & + r^{n-3} & + r^{n-2} & + r^{n-1} & + r^n \\ \hline & r^0 + & 0 + & 0 + & \ldots & + 0 & + 0 & + 0 & - r^n \end{array}$$

$$= \boxed{1 - r^n}$$

Deriving the Geometric sequence (2)

With some basic math,

$$(1-r)(r^0 + r^1 + r^2 + \ldots + r^{n-3} + r^{n-2} + r^{n-1}) = 1 - r^n$$

$$(r^0 + r^1 + r^2 + \ldots + r^{n-3} + r^{n-2} + r^{n-1}) = \frac{1-r^n}{1-r}$$

$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$$

Exercise

Proof that the sum of the geometric sequence using induction.

For all integers $n > 0$,

$$\sum_{i=0}^{n-1} r^i = \frac{1 - r^n}{1 - r}$$

There is a very strong connection between recursion and induction

This connection is fundamental to computer science, both for how we program but also for how we *prove* the correctness of programs and data structures.

Summation as recursion

The sum of 1-to-10, is 10 plus the sum of 1-to-9.

$$\sum_{i=1}^{10} i = \underbrace{1+2+3+4+5+6+7+8+9}_{\sum_{j=1}^{9} j} + 10$$

$$\sum_{i=1}^{10} i = \left(\sum_{j=1}^{9} j\right) + 10$$

And, the sum of 1-to-9, is 9 plus the sum of 1-to-8.

$$\sum_{j=1}^{9} j = \underbrace{1+2+3+4+5+6+7+8}_{\sum_{1=k}^{8} k} + 9$$

$$\sum_{j=1}^{9} j = \left(\sum_{k=1}^{8} k\right) + 9$$

And ...there is a sum that is obvious, $\sum_{i=1}^{1} = 1$

Generalizing the recursion

$$\sum_{i=1}^{n} i = \left(\sum_{j=1}^{n-1} j\right) + n$$

$$\sum_{i=1}^{1} i = 1$$

In terms of functions/programming,

$$\text{Sum}(n) = \text{Sum}(n-1) + n \qquad \text{Sum}(1) = 1$$

Connection between induction and recursion

Recursion: "If I assume the function Sum computes the result for Sum$(n) = x$, then I can compute the result for Sum$(n+1) = x + (n+1)$."

Induction: "If I assume the formula f is correct for $\sum_{i=1}^{n} = \mathrm{f}(n)$, then I can prove that formula f is correct for $\sum_{i=1}^{n+1} = \mathrm{f}(n+1)$."

Additional Exercises

For all integers $n > 0$, $11^n - 4^n$ is divisible by 7

For all integers $n \geq 2$, $n^2 > n + 1$