

Lec 08: Induction I

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The sum of all numbers between 1-100

A folk tale about Gauss is that he was an annoying and disruptive elementary school student, so much so, that his teacher asked him to sum all the numbers between 1-100 to distract him to get some peace and quite in the classroom.


But Gauss found the answer within a few seconds, without having to sum up all the numbers up directly.



Carl Friedrich Gauss
1777-1855

Gauss' "trick"

$$\begin{array}{rcccccccc} & 1 + & 2 + & 3 + & \dots & + 98 & + 99 & + 100 \\ + & 100 + & 99 + & 98 + & \dots & + 3 & + 2 & + 1 \\ \hline & 101 + & 101 + & 101 + & \dots & + 101 & + 101 & + 101 \end{array}$$



$$2 \cdot \sum_{i=1}^{100} = 100 \cdot 101 = 10100$$


Dividing by 2 gives us the result

$$\sum_{i=1}^{100} = (100 \cdot 101)/2 = 5050$$

Generalizing Gauss' "trick"

What if we wanted to find a formula for the sum of the numbers 1-to- n ?

$$\begin{array}{cccccccc} & 1 + & 2 + & 3 + & \dots & + n - 2 & + n - 1 & + n \\ + & n + & n - 1 + & n - 2 + & \dots & + 3 & + 2 & + 1 \\ \hline n + 1 + & n + 1 + & n + 1 + & \dots & + n + 1 & + n + 1 & + n + 1 & + n + 1 \end{array}$$


 n items

$$2 \cdot \sum_{i=1}^n = n(n+1)$$

$$\sum_{i=1}^n = \frac{n(n+1)}{2}$$

Proving this based just on the equation

What if I asked you to prove it based on the equation alone?

$$\forall n \in \mathbb{Z}^+, \sum_1^n = \frac{n(n+1)}{2}$$

We can test for specific values,

$$\sum_{i=1}^3 i = 1 + 2 + 3 = 6 = 3(3-1)/2$$

$$\sum_{i=1}^{10} i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55 = 10(10+1)/2$$

As a universal quantifier, we can't check for all n .

Definition of (Weak) Induction

Definition (Method of Proof by Mathematical Induction)

To prove that $\forall n \geq a$, a property $P(n)$ is true, we must prove two cases:

- **Base Case:** $P(a)$ is true
- **Inductive Step:** $(\forall k \geq a)(P(k) \implies P(k + 1))$ is true.

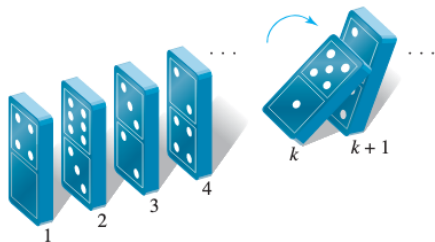


Figure 5.2.3 If the k th domino falls backward, it pushes the $(k + 1)$ st domino backward also.

The Inductive Hypothesis (IH)

Note the inductive step requires that you prove an implication:

$$P(k) \implies P(k + 1)$$

That is, “if $P(k)$, then $P(k + 1)$.”

Because we are proving an implication, we can assume that the hypothesis of the implication, $P(k)$, is true in proving that $P(k + 1)$ is true.

The assumption that $P(k)$ is true is referred to as the **inductive hypothesis**.

Proving Gauss' formula with induction (1)

Theorem

For all integers $n \geq 1$, $\sum_{i=1}^n i = n(n+1)/2$

Let $P(k)$ be the proposition that the sum of the integers 1-to- k is equivalent to the formula $k(k+1)/2$. We must show this to be true $\forall k \geq 1$

We must prove two cases.

Base case: $P(1)$

Inductive case: $(\forall k \geq 1) \left(\underbrace{P(k)}_{\text{Assume IH}} \implies \overbrace{P(k+1)}^{\text{Prove this}} \right)$

Proving Gauss' formula with induction (2)

The base case is straightforward. This is clearly true at $P(1)$.

Proof of Base Case: $P(1) \equiv \sum_{i=1}^1 i = 1(1+1)/2$.

This is directly true because $\sum_{i=1}^1 = 1$ and $1(1+1)/2 = 1$. □

Describing the inductive step

Assume the inductive hypothesis (IH)

$$P(k) \equiv \overbrace{\left[\sum_{i=1}^k i = \frac{k(k+1)}{2} \right]}{\text{Assume this equality is true!}}$$

Can we show?

$$P(k+1) \equiv \underbrace{\left[\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2} \right]}{\text{Can we show this equality is true?}}$$

Inductive step (1)

Proof of Inductive Step: $P(k) \implies P(k+1)$

We can rewrite the sum of the integers formula 1-to- $(n+1)$ with two sums.

$$\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^k i \right) + (k+1)$$

By the inductive hypothesis, $\sum_{i=1}^k i = k(k+1)/2$

$$\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1)$$

Inductive step (2)

Proof of Inductive Step: $P(k) \implies P(k+1)$ (cont.)

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$

Which shows our result. □

Exercise

Proof the following using induction.

$$\sum_{i=1}^n (6i - 2) = n(3n + 1)$$

For all integers $n \geq 1$, $2^n - 1$ is odd.

For all integers $n \geq 1$, $5^n - 1$ is divisible by 4.

Arithmetic and Geometric Sequence

The sum of the first n integers is also called the **arithmetic sum** and comes up often in computer science.

$$\sum_{i=1}^n i \quad \text{Arithmetic Sum}$$

The **geometric sum** is another important summation in computer science that relates to exponentiation of ratios.

$$\sum_{i=0}^{n-1} r^i \quad \text{Geometric Sum}$$

Where $r \in \mathbb{Q}$ (set of rationals).

Deriving the Geometric sequence (1)

Consider the sequence

$$\sum_{i=0}^{n-1} = r^0 + r^1 + r^2 + \dots + r^{n-3} + r^{n-2} + r^{n-1}$$

If we multiply the sequence by $(1 - r)$, it subtracts the sequence from the sequence multiplied by r

$$(1 - r)(r^0 + r^1 + r^2 + \dots + r^{n-3} + r^{n-2} + r^{n-1}) =$$

$$\begin{array}{r} r^0 + r^1 + r^2 + \dots + r^{n-3} + r^{n-2} + r^{n-1} \\ - \quad r^1 + r^2 + \dots + r^{n-3} + r^{n-2} + r^{n-1} + r^n \\ \hline r^0 + 0 + 0 + \dots + 0 + 0 + 0 - r^n \end{array}$$

$$= \boxed{1 - r^n}$$

Deriving the Geometric sequence (2)

With some basic math,

$$(1 - r)(r^0 + r^1 + r^2 + \dots + r^{n-3} + r^{n-2} + r^{n-1}) = 1 - r^n$$

$$(r^0 + r^1 + r^2 + \dots + r^{n-3} + r^{n-2} + r^{n-1}) = \frac{1 - r^n}{1 - r}$$

$$\sum_{i=0}^{n-1} r^i = \frac{1 - r^n}{1 - r}$$

Exercise

Proof that the sum of the geometric sequence using induction.

For all integers $n > 0$,

$$\sum_{i=0}^{n-1} r^i = \frac{1 - r^n}{1 - r}$$

There is a very strong connection between **recursion** and **induction**

This connection is fundamental to computer science, both for how we program but also for how we *prove* the correctness of programs and data structures.

Summation as recursion

The sum of 1-to-10, is 10 plus the sum of 1-to-9.

$$\sum_{i=1}^{10} i = \underbrace{1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9}_{\sum_{j=1}^9 j} + 10$$

$$\sum_{i=1}^{10} i = \left(\sum_{j=1}^9 j \right) + 10$$

And, the sum of 1-to-9, is 9 plus the sum of 1-to-8.

$$\sum_{j=1}^9 j = \underbrace{1 + 2 + 3 + 4 + 5 + 6 + 7 + 8}_{\sum_{k=1}^8 k} + 9$$

$$\sum_{j=1}^9 j = \left(\sum_{k=1}^8 k \right) + 9$$

And ... there is a sum that is obvious, $\sum_{i=1}^1 = 1$

Generalizing the recursion

$$\sum_{i=1}^n i = \left(\sum_{j=1}^{n-1} j \right) + n$$

$$\sum_{i=1}^1 i = 1$$

In terms of functions/programming,

$$\text{Sum}(n) = \text{Sum}(n - 1) + n \quad \text{Sum}(1) = 1$$

Connection between induction and recursion

Recursion: “If I assume the function Sum **computes** the result for $\text{Sum}(n) = x$, then I can **compute** the result for $\text{Sum}(n + 1) = x + (n + 1)$.”

Induction: “If I assume the formula f is **correct** for $\sum_{i=1}^n = f(n)$, then I can **prove** that formula f is **correct** for $\sum_{i=1}^{n+1} = f(n + 1)$.”

Additional Exercises

For all integers $n > 0$, $11^n - 4^n$ is divisible by 7

For all integers $n \geq 2$, $n^2 > n + 1$