Lec 05:
Logical Quantifiers

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Statements

The sentence

A person is a student at GW.

is not a statement. Why?

A statement (or proposition) is a sentence that is either true or false, but not both.

Depends on the value of "A person" – which person?

"John Smith is a student at GW" is a statement.

Predicates (grammatically)

Grammatically, a *predicate* is the part of the sentence that provides information about the subject (or object) of the sentence.

$$\overbrace{\textit{John Smith}}^{\text{subject}}\ \underbrace{\textit{is a student at GW.}}_{\text{predicate}}$$

Predicates (in logic)

A predicate can often be obtained by removing the subject (and other nouns) from the sentence and replacing the noun with a variable.

Let P stand for "x is a student at GW."

Let Q stand for "x is a student at y."

In this example, P and Q are predicate symbols, where $P(x)$ and $Q(x, y)$ are propositional functions with predicate variables x and y.

Predicates, Domains, and Truth Sets

Definition

A predicate is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for those values.

Definition

A domain of a predicate variable is the set of all values that may be substituted for a given predicate variable.

Definition

The truth set of predicate $P(x)$ where x has domain D, is the set of all elements in D that generate true statements. Notationally, the truth set of $P(X)$ is

$$\{x \in D \mid P(X)\}$$

Exercises

Let $P(x)$ be the predicate "$x^3 < x$" with the domain of $\mathbb{R}$, what is

- $P(2)$
- $P(1/2)$
- $P(-2)$

Let $Q(n)$ be the predicate "n is a factor of 12", find the truth set of $Q(n)$ for the following domains:

- $n \in Z^+$
- $n \in Z$

Quantifiers

Definition

A quantifier is a way to convert a predicate into a statement by declaring how many elements (e.g., some, one, all) may be substituted for the predicate to be true, or put another way, define the size of the truth set for a given domain.

Definition

The universal quantifier, denoted with $\forall$ symbol and read "for all", declares that all elements of a given domain satisfy the predicate.

Definition

The extensional quantifier, denoted with $\exists$ symbol and read "exists", declares that there is at least one element of a given domain satisfies the predicate.

The truthfulness of a quantifiers

Consider the predicate $Q(x)$ where x has domain D:

$\forall x \in D, Q(x)$ is true if, and only if, $Q(x)$ is true for every x in D.

$\forall x \in D, Q(x)$ is false if, and only if, $Q(x)$ is false for at least one x in D.

- An x for which $Q(x)$ is false is described as a counter example.

$\exists x \in D, Q(x)$ is true if, and only if, $Q(x)$ is true for at least one x in D.

$\exists x \in D, Q(x)$ is false if, and only if, $Q(x)$ is false for every x in D

Some notes on notation

There is no agreed standard for how to notate quantifiers.

$$\forall x\, P(x) \quad \forall x, \exists y\, Q(x,y)$$

$$(\forall x \in D)P(x)$$

$$(\forall x \in D)(\forall y \in F)(P(x,y))$$

$$(\forall x \in \mathbb{Z})(x > 1 \rightarrow x^2 > x)$$

$$(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(y > x \wedge y > 1 \rightarrow y^2 > x^2)$$

Exercises

Write out the following statements in plain English, and describe if/why the statements are true or false.

$(\forall x \in \mathbb{Z}^+)(x^2 \geq x)$

$(\forall w \in \mathbb{R})(w > 2 \rightarrow w^2 > 4)$

$(\forall q \in \mathbb{Q})(\exists a, b \in \mathbb{Z})(q = \frac{a}{b})$

Reordering Quantifiers

Quantifiers of the same type can be reordered when adjacent.

$$\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$$

$$\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$$

But adjacent quantifiers of different types cannot be reordered.

$$\forall x \exists y P(x, y) \not\equiv \exists y \forall x P(x, y)$$

Ordering of $\forall$ and $\exists$ (example 1)

Consider the preposition $P(x, y)$, where x is homes, y is families, and the predicate is true when family y lives in home x.

$$\forall x \, \exists y \; P(x, y)$$

"For all homes, there exists a family that lives in that home"

$$\exists y \, \forall x \; P(x, y)$$

"There exists a family that lives in all homes"

Ordering of $\forall$ and $\exists$ (example 2)

Consider the preposition $P(x, y)$, where x is "I take the metro" homes, y is "a person is stabbed", and the predicate when someone is stabbed on the metro.

$$\forall x\, \exists y\, P(x, y)$$

"For all times I take metro, there exists a person who is stabbed."

$$\exists y\, \forall x\, P(x, y)$$

"There exists a person who is stabbed every time I ride metro."

Exercise

Write the following quantifiers in plain English

$(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x > y)$

$(\exists y \in \mathbb{Z})(\forall x \in \mathbb{Z})(x > y)$

$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\exists a, b \in \mathbb{R})(x = 2/a \wedge y = 2/b)$

$(\exists y \in \mathbb{Z})(\exists x \in \mathbb{Z})(2x = y)$

Implicit Quantification

We can write a statement with an implicit quantification, easily and naturally.

$$\text{if } x > 2 \text{ then } x^2 > 4$$

This is equivalent to

$$\forall x \in \mathbb{R}, \text{ if } x > 2 \text{ then } x^2 > 4$$

To indicate an explicit quantification we can use $\implies$ and $\iff$ between predicates:

$$x > 2 \implies x^2 > 4$$

The domain of the predicates is assumed to be the same (unless otherwise indicated)

Predicative conditional and bi-conditional

Definition

For predicates $P(x)$ and $Q(x)$ with the same domain U, and P^t and Q^t are the truth set of P and Q,

- $P(x) \implies Q(x) \equiv (\forall x \in U)(P(x) \rightarrow Q(x))$
  - Every element of P's truth set is in Q's truth set ($P^t \subseteq Q^t$)
- $P(x) \iff Q(x) \equiv (\forall x \in U)(P(x) \leftrightarrow Q(x))$
  - The truth sets of P and Q are equivalent ($P^t \subseteq Q^t \wedge Q^t \subseteq P^t$)

* We may use and $\implies$ over $\rightarrow$ in some cases, despite there not always being an implicit quantification.

Quantifiers in a Finite Domains

If we consider the domain of a predicate to be finite
$D = \{x_1, x_2, \ldots, x_n\}$
How can we rewrite the following?

$(\forall x \in D)(P(x))$

- $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$
- All finite values must be true for the universal quantifier to be true

$(\exists x \in D)(P(X))$

- $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$
- At least one of the finite values must be true for the extensional quantifier to be true

Negation of a Universal Quantifier

What is the inverse of the follow statement?

$$(\forall x \in D)(P(x))$$

Negation of a statement that is true when the statement is false. A forall quantifier is false when there exists a counter example.

$$(\exists x \in D)(\neg P(x))$$

Negation of an Existential Quantifier

What is the inverse of the following statement?

$$(\exists x \in D)(P(x))$$

An existential is false when there *does not exist* an example, or *for all examples* the predicate is false.

$$(\forall x \in D)(\neg P(x))$$

Negation of Universal Conditional Statement

What is the inverse of the follow statement?

$$(\forall x \in U)(P(x) \rightarrow Q(x))$$

$$\neg[(\forall x \in U)(P(x) \rightarrow Q(x)] \equiv$$
$$(\exists x \in U)[\neg(P(x) \rightarrow Q(x))]$$

Recall that $\neg(p \rightarrow q) \equiv p \wedge \neg q$

$$(\exists x \in U)[\neg(P(x) \rightarrow Q(x))] \equiv (\exists x \in U)(P(x) \wedge \neg Q(x))$$

Connection to DeMorgan's Law

DeMorgan's Law shows us that the inversion of a $\wedge$ is a $\vee$, and the inversion of a $\vee$ is a $\wedge$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Similarly, the inversion of $\forall$ is an $\exists$, and the inversion of a $\exists$ is a $\forall$.

$$\neg(\forall x, P(x)) \equiv \exists x, \neg P(x)$$
$$\neg(\exists x, P(x)) \equiv \forall x, \neg P(x)$$

Exercises

Write a logically equivalent statement for the following English sentences

There is at least three distinct integers for which $P(x)$ is true.

There is at most three distinct integers for which $P(x)$ is true.

There is exactly three distinct integers for which the $P(x)$ is true.

There does not exist two distinct integers for which the $P(x)$ is true.