

Lec 04: Logical Implications

Prof. Adam J. Aviv

GW

CSCI 1311 Discrete Structures I
Spring 2023

Conditional Statement

If $\overbrace{\text{a person is a student at GW}}^p$ *and* $\overbrace{\text{a computer science major}}^q$,
then $\underbrace{\text{the student will take CSCI 1311.}}_r$

This sentence is a conditional statement because the truth of the outcome r is **condition** on the truth of the condition $p \wedge q$

Implication \rightarrow

Definition

If p and q are statement variables, the **conditional** of q by p (read “If p then q ” or “ p implies q ” and written $p \rightarrow q$) is false whenever p is true and q is false, and true otherwise.

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Definition

In an implication, $p \rightarrow q$, we describe p as the **hypothesis** (or **antecedent**) of the conditional and q as the **conclusion** (or **consequent**).

Implications in English: $p \rightarrow q$

If you are a CS major, then you have to take Discrete Math.

- false \rightarrow false
You are not a CS Major, you do not have to take Discrete Math.
 - ▶ **True.** You do not have to take Discrete Math if you are not a CS major.
- false \rightarrow true
You are not a CS major, you do have to take Discrete Math.
 - ▶ **True.** Some non-CS majors have to take discrete math.

The conclusion can be true even when they hypothesis is false.
- true \rightarrow true
You are a CS Major, you do have to take Discrete Math.
 - ▶ **True.** All CS majors do take discrete math.
- true \rightarrow false
You are a CS Major, you do not have to take Discrete Math.
 - ▶ **False.** A CS major must take Discrete Math.

The conclusion cannot be false when they hypothesis is true.

Vacuous Truths

Definition

A conditional statement that is true by virtue of the fact that its hypothesis is false is described as **vacuously true** or **true by default**

Example

The conditional statement

If you are a CS major, then you have to take Discrete Math.

is **vacuously true** if you are taking Discrete Math but you are not a CS major. The hypothesis that you are CS major may be false, but the conclusion is still true.

Implication Truth Table

$$p \rightarrow \neg q$$

Hyp. \underbrace{p}	q	Conc. $\underbrace{\neg q}$	$p \rightarrow \neg q$
F	F	T	T
F	T	F	T
T	F	T	T
T	T	F	F

Exercises

Construct a truth table for the following implications:

$$\neg p \rightarrow q$$

$$p \vee \neg q \rightarrow \neg q$$

Note that \rightarrow order of precedence is the lowest: $()$, \neg , \wedge , \vee and then \rightarrow

Logical equivalences with \rightarrow

We can show equivalences with \rightarrow using truth tables, as it is like other logical operators, for example.

$$p \vee q \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$$

p	q	r	$p \vee q$	$p \vee q \rightarrow r$	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$
F	F	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	T	F	T	F	T	F	F
F	T	T	T	T	T	T	T
T	F	F	T	F	F	T	F
T	F	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	T	T	T	T	T	T	T

Can you convince yourself in plain English why this equivalence is true?

Logical equivalent statement for \rightarrow

But, anything we can prove with a truth table, we should be able to show via equivalent statements.

Consider the truth table for $p \rightarrow q$

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Can we write a statement to represent implication without \rightarrow ?

Logical equivalent statement for \rightarrow

$p \rightarrow q$, is false when p is true and q is false, and true otherwise.

$$\overbrace{\neg(p \wedge \neg q)}^{\text{negated gives us all true cases}} \equiv p \rightarrow q$$

Test for false case

Further simplified using De Morgan's law

$$\neg p \vee q \equiv p \rightarrow q$$

Negating an Implication

What is a logically equivalent statement to

$$\neg(p \rightarrow q)$$

$p \rightarrow q$ is false when q is true and p is false, and true otherwise. So $\neg(p \rightarrow q)$ is false when q is true p is false, and true otherwise.

$$\neg(p \rightarrow q) \equiv \underbrace{\neg q \wedge p}_{\text{Test for the false case}}$$

Exercise

Write the equivalent logical statement for the following implications, without \rightarrow . Recall that $p \rightarrow q = \neg p \vee q$. Try and simplify as much as possible.

$$p \wedge q \rightarrow \neg r$$

$$\neg q \rightarrow \neg p$$

Contrapositive

Definition

The **contrapositive** of a conditional statement, "if p , then q " is "if $\neg q$, then $\neg p$."

A conditional statement is equivalent to its contrapositive.

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Only-if

Consider the phrase:

$\overbrace{\text{You are a CS major}}^p$ only if $\underbrace{\text{you take discrete math}}_q$

p can take places **only if** q takes places, or rephrased, if q does not take place, then p cannot take place.

$$\neg q \rightarrow \neg p$$

That's just the contrapositive!

$$\neg q \rightarrow \neg p \equiv p \rightarrow q$$

Definition

The statement p **only if** q is equivalent to $p \rightarrow q$

Biconditional

Suppose we have an implication in both directions,

$\overbrace{\text{You are a CS major}}^p$ only if $\underbrace{\text{you take discrete math}}_q$

and

$\underbrace{\text{You take discrete math}}_q$ only if $\overbrace{\text{you are a CS major}}^p$

In logic, this is written

$$(p \rightarrow q) \wedge (q \rightarrow p)$$

and described as a **biconditional**.

If-and-only-if \leftrightarrow

A biconditional can be rephrased as an "if and only if" statement

$\overbrace{\text{You are a CS major}}^p$ if and only if $\underbrace{\text{you take discrete math}}_q$

Written $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	F
T	T	T	T	T

Sufficient Conditions

We can also write the phrase

$\overbrace{\text{Being a CS major}}^p$ is a sufficient condition $\underbrace{\text{to take discrete math}}_q$

In other words, p being true is a sufficient for it to be the case that q is true, but q could be true without p .

This is the same as "if p , then q " or $p \rightarrow q$.

Necessary Conditions

If we were to write the following

$\overbrace{\text{Being a CS major}}^p$ is a necessary condition $\underbrace{\text{to take discrete math}}_q$

In other words, p must be true for q to be true, or put another way, if p is not true (or false) then q must be false.

This is the same as "if not p , then not q " or $\neg p \rightarrow \neg q$.

Necessary and Sufficient Condition

If we were to write the following

$\overbrace{\text{Being a CS major is a necessary and sufficient condition to}}^p \underbrace{\text{take discrete math}}_q$

We are saying both “if p , then q ” and “if not p , then not q ”

$$(p \rightarrow q) \wedge (\neg p \rightarrow \neg q)$$

by the contrapositive

$$(p \rightarrow q) \wedge (q \rightarrow p) \equiv p \leftrightarrow q$$

A necessary and sufficient condition is an if-and-only-if statement.

Exercises

Convert the following plain language sentences into logical statements.

- (A) *Having a BS or BE is a sufficient condition for having a bachelors degree.*
- (B) *A person turns 10 years old today if and only if that persons birthday is today and it was 10 years ago.*
- (C) *If 10 people are in a car and the car is small then it is a clown car.*
- (D) *It is necessary to write term papers to pass a history class*
- (E) *Taking all the requires classes and submitting for a CS degree is a necessary and sufficient for earning a CS degree.*