Lec 04: Logical Implications

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GW

CSCI 1311 Discrete Structures I Spring 2023

Conditional Statement

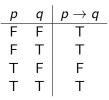


This sentence is a conditional statement because the truth of the outcome r is condition on the truth of the condition $p \land q$

Implication \rightarrow

Definition

If p and q are statement variables, the conditional of q by p (read "If p then q" or "p implies q" and written $p \rightarrow q$) is false whenever p is true and q is false, and true otherwise.



Definition

In an implication, $p \rightarrow q$, we describe p as the hypothesis (or antecedent) of the conditional and q as the conclusion (or consequent).

Implications in English: $p \rightarrow q$

If you are a CS major, then you have to take Discrete Math.

• false \rightarrow false

You are not a CS Major, you do not have take Discrete Math.

- **True**. You do not have to take Discrete Math if you are not a CS major.
- false → true You are not a CS major, you do have to take Discrete Math.
 - **True**. Some non-CS majors have to take discrete math.

The conclusion can be true even when they hypothesis is false.

• true \rightarrow true

You are a CS Major, you do have to take Discrete Math.

- True. All CS majors do take discrete math.
- true → false
 You are a CS Major, you do not have to take Discrete Math.
 - False. A CS major must take Discrete Math.

The conclusion cannot be false when they hypothesis is true.

Vacuous Truths

Definition

A conditional statement that is true by virtue of the fact that its hypothesis is false is described as vacuously true or true by default

Example

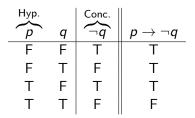
The conditional statement

If you are a CS major, then you have to take Discrete Math.

is vacuously true If you are taking Discrete Math but you are not a CS major. The hypothesis that you are CS major may be false, but the conclusion is still true.

Implication Truth Table

$$p
ightarrow \neg q$$



Exercises

Construct a truth table for the following implications:

eg p
ightarrow q

 $p \lor \neg q \to \neg q$

Note that \rightarrow order of precedence is the lowest: (), \neg, \wedge, \vee and then \rightarrow

Logical equivalences with \rightarrow

We can show equivalences with \rightarrow using truth tables, as it is like other logical operators, for example.

$p \lor q ightarrow r \equiv (p ightarrow r) \land (q ightarrow r)$							
р	q	r	$p \lor q$	$p \lor q \to r$	p ightarrow r	q ightarrow r	$ (p ightarrow r) \wedge (q ightarrow r)$
F	F	F	F	Т	Т	Т	Т
F	F	Т	F	Т	Т	Т	Т
F	Т	F	Т	F	Т	F	F
F	Т	Т	Т	Т	Т	Т	Т
Т	F	F	Т	F	F	Т	F
Т	F	Т	Т	Т	Т	Т	Т
Т	Т	F	Т	F	F	F	F
Т	Т	Т	Т	Т	Т	Т	Т

Can you convince yourself in plain English why this equivalence is true?

Logical equivalent statement for \rightarrow

But, anything we can prove with a truth table, we should be able to show via equivalent statements.

Consider the truth table for p
ightarrow q

$$\begin{array}{c|cc} p & q & p \rightarrow q \\ \hline F & F & T \\ F & T & T \\ T & F & F \\ T & T & T \end{array}$$

Can we write a statement to represent implication without ightarrow ?

Logical equivalent statement for \rightarrow

p
ightarrow q, is false when p is true and q is false, and true otherwise.

negated gives us all true cases
$$\neg(\underbrace{p \land \neg q}_{\text{Test for false case}}) \equiv p \rightarrow q$$

Further simplified using De Morgan's law

$$\neg p \lor q \equiv p \to q$$

Negating an Implication

What is a logically equivalent statement to

 $p \rightarrow q$ is false when q is true and p is false, and true otherwise So $\neg(p \rightarrow q)$ is false when q is true p is false, and false otherwise.

$$eg (p o q) \equiv \underbrace{
eg q \wedge p}_{ ext{Test for the false case}}$$

Exercise

Write the equivalent logical statement for the following implications, without \rightarrow . Recall that $p \rightarrow q = \neg p \lor q$. Try and simplify as much as possible.

 $p \land q \rightarrow \neg r$

 $\neg q \rightarrow \neg p$

Contrapositive

Definition

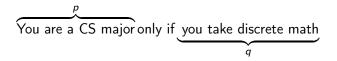
The contrapositive of a conditional statement, "if p, then q" is "if $\neg q$, then $\neg p$."

A conditional statement is equivalent to its contrapositive.

$$p
ightarrow q \equiv \neg q
ightarrow \neg p$$

Only-if

Consider the phrase:



p can take places only if q takes places, or rephrased, if q does not take place, then p cannot take place.

$$\neg q
ightarrow \neg p$$

That's just the contrapositive!

$$\neg q \rightarrow \neg p \equiv p \rightarrow q$$

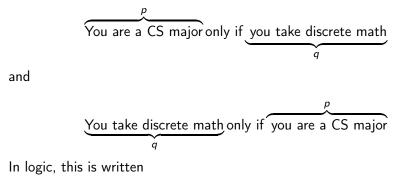
Definition

The statement p only if q is equivalent to $p \rightarrow q$

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Biconditional

Suppose we have an implication in both directions,



$$(p
ightarrow q) \wedge (q
ightarrow p)$$

and described as a biconditional.

$\textbf{If-and-only-if} \leftrightarrow$

A biconditional can be rephrased as an "if and only if" statement

$$\overbrace{\text{You are a CS major if and only if you take discrete math}}^{p}$$
Written $p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$

$$p$$
 q $p \rightarrow q$ $q \rightarrow p$ $p \leftrightarrow q$ FFTTTFTTFFTFFTFTTTTT

Sufficient Conditions

We can also write the phrase

$$\underbrace{\mathsf{Being a CS major is a sufficient condition to take discrete math}_{q}}_{q}$$

In other words, p being true is a sufficient for it to be the case that q is true, but q could be true without p.

This is the same as "if p, then q" or $p \rightarrow q$.

Necessary Conditions

If we were to write the following

$$\underbrace{\overset{p}{\text{Being a CS major is a necessary condition to take discrete math}}_{q}$$

In other words, p must be true for q to be true, or put another way, if p is not true (or false) then q must be false.

This is the same as "if not p, then not q" or $\neg p \rightarrow \neg q$.

Necessary and Sufficient Condition

If we were to write the following

Being a CS major is a necessary and sufficient condition to take discrete math $\frac{p}{q}$

We are saying both "if p, then q" and "if not p, then not q"

$$(p
ightarrow q) \wedge (\neg p
ightarrow \neg q)$$

by the contrapositive

$$(p
ightarrow q) \wedge (q
ightarrow p) \quad \equiv \quad p \leftrightarrow q$$

A necessary and sufficient condition is an if-and-only-if statement.

Exercises

Convert the following plain language sentences into logical statements.

(A) Having a BS or BE is a sufficient condition for having a bachelors degree.

(B) A person turns 10 years old today if and only if that persons birthday is today and it was 10 years ago.

(C) If 10 people are in a car and the car is small then it is a clown car.

(D) It is necessary to write term papers to pass a history class

(E) Taking all the requires classes and submitting for a CS degree is a necessary and sufficient for earning a CS degree.