Lec 04:
Logical Implications

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GW

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Conditional Statement

If $\overbrace{\textit{a person is a student at GW}}^{p}$ *and* $\overbrace{\textit{a computer science major}}^{q}$*,*
then $\underbrace{\textit{the student will take CSCI 1311}}_{r}$*.*

This sentence is a conditional statement because the truth of the outcome r is condition on the truth of the condition $p \wedge q$

Implication $\rightarrow$

Definition

If p and q are statement variables, the conditional of q by p (read "If p then q" or "p implies q" and written $p \rightarrow q$) is false whenever p is true and q is false, and true otherwise.

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Definition

In an implication, $p \rightarrow q$, we describe p as the hypothesis (or antecedent) of the conditional and q as the conclusion (or consequent).

Implications in English: $p \rightarrow q$

If you are a CS major, then you have to take Discrete Math.

- false $\rightarrow$ false
  You are not a CS Major, you do not have take Discrete Math.
  - True. You do not have to take Discrete Math if you are not a CS major.
- false $\rightarrow$ true
  You are not a CS major, you do have to take Discrete Math.
  - True. Some non-CS majors have to take discrete math.

    The conclusion can be true even when they hypothesis is false.
- true $\rightarrow$ true
  You are a CS Major, you do have to take Discrete Math.
  - True. All CS majors do take discrete math.
- true $\rightarrow$ false
  You are a CS Major, you do not have to take Discrete Math.
  - False. A CS major must take Discrete Math.

    The conclusion cannot be false when they hypothesis is true.

Vacuous Truths

Definition

A conditional statement that is true by virtue of the fact that its hypothesis is false is described as vacuously true or true by default

Example

The conditional statement

If you are a CS major, then you have to take Discrete Math.

is vacuously true If you are taking Discrete Math but you are not a CS major. The hypothesis that you are CS major may be false, but the conclusion is still true.

Implication Truth Table

$$p \rightarrow \neg q$$

$\overbrace{p}^{\text{Hyp.}}$	q	$\overbrace{\neg q}^{\text{Conc.}}$	$p \rightarrow \neg q$
F	F	T	T
F	T	F	T
T	F	T	T
T	T	F	F

Exercises

Construct a truth table for the following implications:

$\neg p \rightarrow q$

$p \vee \neg q \rightarrow \neg q$

Note that $\rightarrow$ order of precedence is the lowest: $(), \neg, \wedge, \vee$ and then $\rightarrow$

Logical equivalences with $\rightarrow$

We can show equivalences with $\rightarrow$ using truth tables, as it is like other logical operators, for example.

$$p \vee q \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$$

p	q	r	$p \vee q$	$p \vee q \rightarrow r$	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$
F	F	F	F	**T**	T	T	**T**
F	F	T	F	**T**	T	T	**T**
F	T	F	T	**F**	T	F	**F**
F	T	T	T	**T**	T	T	**T**
T	F	F	T	**F**	F	T	**F**
T	F	T	T	**T**	T	T	**T**
T	T	F	T	**F**	F	F	**F**
T	T	T	T	**T**	T	T	**T**

Can you convince yourself in plain English why this equivalence is true?

Logical equivalent statement for $\rightarrow$

But, anything we can prove with a truth table, we should be able to show via equivalent statements.

Consider the truth table for $p \rightarrow q$

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Can we write a statement to represent implication without $\rightarrow$?

Logical equivalent statement for $\rightarrow$

$p \rightarrow q$, is false when p is true and q is false, and true otherwise.

$$\overbrace{\neg(\quad \underbrace{p \land \neg q}_{\text{Test for false case}} \quad)}^{\text{negated gives us all true cases}} \equiv p \rightarrow q$$

Further simplified using De Morgan's law

$$\neg p \lor q \equiv p \rightarrow q$$

Negating an Implication

What is a logically equivalent statement to

$$\neg(p \rightarrow q)$$

$p \rightarrow q$ is false when q is true and p is false, and true otherwise So $\neg(p \rightarrow q)$ is false when q is true p is false, and false otherwise.

$$\neg(p \rightarrow q) \equiv \underbrace{\neg q \wedge p}_{\text{Test for the false case}}$$

Exercise

Write the equivalent logical statement for the following implications, without $\rightarrow$. Recall that $p \rightarrow q = \neg p \vee q$. Try and simplify as much as possible.

$p \wedge q \rightarrow \neg r$

$\neg q \rightarrow \neg p$

Contrapositive

Definition

The contrapositive of a conditional statement,
"if p, then q" is "if $\neg q$, then $\neg p$."

A conditional statement is equivalent to its contrapositive.

$$p \rightarrow q \quad \equiv \quad \neg q \rightarrow \neg p$$

Only-if

Consider the phrase:

$$\overbrace{\text{You are a CS major}}^{p}\text{ only if }\underbrace{\text{you take discrete math}}_{q}$$

p can take places only if q takes places, or rephrased, if q does not take place, then p cannot take place.

$$\neg q \rightarrow \neg p$$

That's just the contrapositive!

$$\neg q \rightarrow \neg p \equiv p \rightarrow q$$

Definition

The statement p only if q is equivalent to $p \rightarrow q$

Biconditional

Suppose we have an implication in both directions,

$$\overbrace{\text{You are a CS major}}^{p} \text{ only if } \underbrace{\text{you take discrete math}}_{q}$$

and

$$\underbrace{\text{You take discrete math}}_{q} \text{ only if } \overbrace{\text{you are a CS major}}^{p}$$

In logic, this is written

$$(p \to q) \wedge (q \to p)$$

and described as a biconditional.

If-and-only-if $\leftrightarrow$

A biconditional can be rephrased as an "if and only if" statement

$$\overbrace{\text{You are a CS major}}^{p}\ \text{if and only if}\ \underbrace{\text{you take discrete math}}_{q}$$

Written $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	F
T	T	T	T	T

Sufficient Conditions

We can also write the phrase

$$\overbrace{\text{Being a CS major}}^{p}\text{ is a sufficient condition }\underbrace{\text{to take discrete math}}_{q}$$

In other words, p being true is a sufficient for it to be the case that q is true, but q could be true without p.

This is the same as "if p, then q" or $p \rightarrow q$.

Necessary Conditions

If we were to write the following

$$\overbrace{\text{Being a CS major}}^{p}\text{ is a necessary condition }\underbrace{\text{to take discrete math}}_{q}$$

In other words, p must be true for q to be true, or put another way, if p is not true (or false) then q must be false.

This is the same as "if not p, then not q" or $\neg p \rightarrow \neg q$.

Necessary and Sufficient Condition

If we were to write the following

$\overbrace{\text{Being a CS major}}^{p}$ is a necessary and sufficient condition $\underbrace{\text{to take discrete math}}_{q}$

We are saying both "if p, then q" and "if not p, then not q"

$$(p \rightarrow q) \wedge (\neg p \rightarrow \neg q)$$

by the contrapositive

$$(p \rightarrow q) \wedge (q \rightarrow p) \quad \equiv \quad p \leftrightarrow q$$

A necessary and sufficient condition is an if-and-only-if statement.

Exercises

Convert the following plain language sentences into logical statements.

(A) *Having a BS or BE is a sufficient condition for having a bachelors degree.*

(B) *A person turns 10 years old today if and only if that persons birthday is today and it was 10 years ago.*

(C) *If 10 people are in a car and the car is small then it is a clown car.*

(D) *It is necessary to write term papers to pass a history class*

(E) *Taking all the requires classes and submitting for a CS degree is a necessary and sufficient for earning a CS degree.*