Lec 03:
Propositional Logic I

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Logic

We use logic as the basic framework for how to show (or disprove) that a proposition (or statement) is either true (or false).

Example

Consider the logic of the following statements.

- *If a person is a student at GW and a computer science major, then the student will take CSCI 1311.*
- *John Smith is a student at GW.*
- *John Smith is a computer science major.*

Therefor we can deduce that Arthur will take CSCI 1311 as a true statement.

Propositions and Statement

Definition

A proposition is a statement that is either true or false.

Example

Some statements that are propositions:

- "John Smith is a student at GW" – that can be checked!
- $\sqrt{2} \notin \mathbb{Q}$ – We can prove this!
- $1 + 1 = 3$ – This is obviously false!

Example

But not all statements are prepositions:

- "He is a student at GW" – Whose he?
- $2 + 2$ – is just an expression?!
- $x^2 + 3x = 5$ – which x?
- $\mathbb{Z}^+$ – this is just a set!

Compound Statement

We can combine singular propositions into compound statement that relies on the truthfulness of the individual parts.

Example

The following proposition is a compound statement:

> *John Smith is a student at GW and John Smith is a computer science major.*

For the statement to be true, both

- *John Smith is a student at GW*
- *John Smith is a computer science major*

must be individually true statements.

Symbols of Compound Statements

Consider two propositions, p and q, we can define the following logical operations to produce compound statements

- Conjunction: $p \wedge q$ ("p and q").
  True only when both p and q are true.
- Disjunction: $p \vee q$ ("p or q").
  True when either p or q are true.
- Negation: $\neg p$ ("not p").
  True when p is false.*

Statements like the one above, using variables, are referred to as propositional forms.

* Note that there are many ways to denote negation. The book uses $\sim p$. You may see the following notation for negation $\overline{p}$ elsewhere. We will try and use the $\neg$ symbol wherever possible for consistency.

Truth Tables

One way to understand a statements is based on all possible truth values for composite propositions. We express this using a truth table.

p	$\neg p$
F	T
T	F

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

Truth Tables of Compound Statements

We can do truth tables for any logical statement

$$p \vee (q \wedge r)$$

Order of Operations

Like in arithmetic, we need to obey an order of operations for logical operators. For example, how would we evaluate

$$p \vee \neg q \wedge r$$

Operationally:

1. $()$
2. $\neg$
3. $\wedge$
4. $\vee$

Exercise

Draw the truth tables for the following statements

$\neg p \vee q$

$p \vee q \wedge \neg r$

$\neg r \wedge (p \vee \neg q)$

Tautologies and Contradictions

What are the truth table for the following statements?

$p \wedge \neg p$

$p \vee \neg p$

Tautologies and Contradictions

Definition

A statement that is always true, regardless of the truth values, is called a tautology, written as t, while a statement that is always false, regardless of the truth values, is a contradiction, written as c.

$$p \wedge \mathrm{t} = p \qquad p \vee \mathrm{t} = \mathrm{t}$$

$$p \wedge \mathrm{c} = \mathrm{c} \qquad p \vee \mathrm{c} = p$$

Exclusive Or (XOR)

Another common operator is exclusive or, written as XOR or $\oplus$. The XOR of p and q is true when either p or q are true, but not when both p and q are true.

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	F

Note: The precedence order for $\oplus$ comes before $\vee$ but after $\wedge$,
so $\neg p \oplus q \wedge r$ is grouped $(\neg p) \oplus (q \wedge r)$,

while $p \oplus q \vee r$ is grouped $(p \oplus q) \vee r$

XOR as a compound statement

The operators $\neg, \vee, \wedge$ is functionally complete[1], which means all binary operators in Boolean logic can be expressed in terms of those operators, including XOR.

Recall the English statement describing XOR:

> *The XOR of p and q is true when either p or q are true, but not when both p and q are true.*

Can you derive a compound logical statement that is equivalent?

[1] But it is not the minimal set of operators. You actually only need $\neg$ and one of $\{\vee, \wedge, \rightarrow\}$ to be functionally complete.

Logical equivalency via truth values

Definition

Two statements are logically equivalent, written $P \equiv Q$, if and only if, they have equivalent truth values for each possible substitution of statement variables (that is, they have the same truth table).

Can we prove the following equivalency?

$$p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$$

Proof of equivalency via truth table

$$p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$$

p	q	$p \oplus q$
F	F	**F**
F	T	**T**
T	F	**T**
T	T	**F**

p	q	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$
F	F	F	F	T	**F**
F	T	T	F	T	**T**
T	F	T	F	T	**T**
T	T	T	T	F	**F**

Exercises

Using a truth table, describe the following equivalencies as true or false

$\neg(p \wedge q) \equiv \neg p \wedge \neg q$

$(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$

Logical equivalency via equivalent forms

Definition

Two statements are logically equivalent, written $P \equiv Q$, if and only if, they have equivalent forms when identical component statements are used to replace identical component statements.

That is, we can show an equivalency through substation of other equivalent statements. For example, the same we can show the following:

$$\begin{aligned} 2+4+8 &= 10+4 & \\ 2+8+4 &= 10+4 & \textit{by commutativity of addition} \\ 10+4 &= 10+4 & \textit{by addition of 8 and 2} \end{aligned}$$

A logically equivalent form of XOR

Recall the truth table for XOR

p	q	$\mathrm{p} \oplus \mathrm{q}$	$(\mathrm{p} \wedge \neg \mathrm{q}) \vee (\neg \mathrm{p} \wedge \mathrm{q})$	$(p \wedge \neg q)$	$(\neg p \wedge q)$
F	F	**F**	**F**	F	F
F	T	**T**	**T**	F	T
T	F	**T**	**T**	T	F
T	T	**F**	**F**	F	F

From which, we can see that another way to describe XOR as

$$(p \wedge \neg q) \vee (\neg p \wedge q)$$

"Either p is true and q is false, or p is false and q is true"

Proving the logically equivalent form of XOR

We should be able to find a way to convert one propositional statement into the other?

$$(p \land \neg q) \lor (\neg p \land q) \equiv (p \lor q) \land \underbrace{\neg(p \land q)}_{?}$$

First, we will need to understand how to handle negations.

Negations of an AND/OR statment

Consider the statements:

> *John Smith is a student and John Smith is a computer science major*
>
> *John Smith is a student or John Smith is a computer science major*

What are the negation?

> *John Smith is not a student OR John Smith is not a computer science major.*
>
> *John Smith is not a student AND John Smith is not a computer science major.*

De Morgan's Law

Definition

De Morgan's law states that

- the negation of an *and statement* is logically equivalent to an *or statement* with each of its components negated, and
- the negation of an *or statement* is logically equivalent to an *or statement* with each of its components negated.

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Can you show this via a truth table? (Check it yourself later)

Other logical equivalencies

Let t be a tautology (i.e., true) and c be a contradiction (i.e., false), and p, q, and r be propositions.

Commutative Law	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative Law	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive Law	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity law	$p \wedge \mathrm{t} \equiv p$	$p \vee \mathrm{c} \equiv p$
Negation law	$p \vee \neg p \equiv \mathrm{t}$	$p \wedge \neg p \equiv \mathrm{c}$
Double negation law	$\neg(\neg p) \equiv p$	
Idempotent law	$p \wedge p \equiv p$	$p \vee p \equiv p$
Universal bound law	$p \vee \mathrm{t} \equiv \mathrm{t}$	$p \wedge \mathrm{c} \equiv \mathrm{c}$
De Morgan's law	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
Absorption law	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negations of t *and* c	$\neg \mathrm{t} \equiv \mathrm{c}$	$\neg \mathrm{c} \equiv \mathrm{t}$

Showing logical equivalence of XOR statements

$$
\begin{aligned}
(p \wedge \neg q) \vee (\neg p \wedge q) &\equiv (p \vee q) \wedge \neg(p \wedge q) && \\
&\equiv (p \vee q) \wedge (\neg p \vee \neg q) && \textit{by De Morgan} \\
&\equiv ((p \vee q) \wedge \neg p) \vee ((p \vee q) \wedge \neg q) && \textit{by Distributive} \\
&\equiv ((p \wedge \neg p) \vee (q \wedge \neg p)) \quad \vee && \textit{by Distributive} \\
&\quad\; ((p \wedge \neg q) \vee (q \wedge \neg q)) && \\
&\equiv (\mathrm{c} \vee (q \wedge \neg p) \vee ((p \wedge \neg q) \vee \mathrm{c}) && \textit{by Negation} \\
&\equiv (q \wedge \neg p) \vee (p \wedge \neg q) && \textit{by Identity} \\
&\equiv (p \wedge \neg q) \vee (q \wedge \neg p) && \textit{by Commutative} \\
&\equiv (p \wedge \neg q) \vee (\neg p \wedge q) && \textit{by Commutative}
\end{aligned}
$$

Exercise

Commutative Law	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative Law	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive Law	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity law	$p \wedge \mathrm{t} \equiv p$	$p \vee \mathrm{c} \equiv p$
Negation law	$p \vee \neg p \equiv \mathrm{t}$	$p \wedge \neg p \equiv \mathrm{c}$
Double negation law	$\neg(\neg p) \equiv p$	
Idempotent law	$p \wedge p \equiv p$	$p \vee p \equiv p$
Universal bound law	$p \vee \mathrm{t} \equiv \mathrm{t}$	$p \wedge \mathrm{c} \equiv \mathrm{c}$
De Morgan's law	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
Absorption law	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negations of t *and* c	$\neg \mathrm{t} \equiv \mathrm{c}$	$\neg \mathrm{c} \equiv \mathrm{t}$

Show the logical equivalence via equivalent statements

$$\neg r \vee p \vee q \vee (p \wedge \neg r) \equiv \neg(\neg p \wedge \neg q \wedge r)$$